THEORIES WITHOUT THE TREE PROPERTY OF THE SECOND KIND

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Abstract. We initiate a systematic study of the class of theories without the tree property of the second kind — NTP_2. Most importantly, we show: the burden is "sub-multiplicative" in arbitrary theories (in particular, if a theory has TP_2 then there is a formula with a single variable witnessing this); NTP_2 is equivalent to the generalized Kim's lemma; the dp-rank of a type in an arbitrary theory is witnessed by mutually indiscernible sequences of realizations of the type, after adding some parameters — so the dp-rank of a 1-type in any theory is always witnessed by sequences of singletons; in NTP_2 theories, simple types are co-simple, characterized by the co-independence theorem, and forking between the realizations of a simple type and arbitrary elements satisfies full symmetry; a Henselian valued field of characteristic $(0,0)$ is NTP_2 (strong, of finite burden) if and only if the residue field is NTP_2 (the residue field and the value group are strong, of finite burden respectively); adding a generic predicate to a geometric NTP_2 theory preserves NTP_2.

Introduction

The aim of this paper is to initiate a systematic study of theories without the tree property of the second kind, or NTP_2 theories. This class was defined by Shelah implicitly in [She90] in terms of a certain cardinal invariant κ_{inp} (see Section 1) and explicitly in [She80], and it contains both simple and NIP theories. There was no active research on the subject until the recent interest in generalizing methods and results of stability theory to larger contexts, necessitated for example by the developments in the model theory of important algebraic examples such as algebraically closed valued fields [HHM08].

We give a short overview of related results in the literature. The invariant κ_{inp}, an upper bound for the number of independent partitions, was considered by Tsuboi in [Tsu85] for the case of stable theories. In [Adl08] Adler defines burden, by relativizing κ_{inp} to a fixed partial type, makes the connection to weight in simple theories and defines strong theories. Burden in the context of NIP theories, where it is called dp-rank, was already introduced by Shelah in [She05] and developed further in [OU11]. Results about forking and dividing in NTP_2 theories were established in [CK12]. In particular, it was proved that a formula forks over a model if and only if it divides over it (see Section 3). Some facts about ordered inp-minimal theories and groups (that is with $\kappa^1_{\mathrm{inp}} = 1$) are proved in [Goo10, Sim11]. In [BY11, Theorem 4.13] Ben Yaacov shows that if a structure has

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IP, then its randomization (in the sense of continuous logic) has TP_2. Malliaris [Mal12] considers TP_2 in relation to the saturation of ultra-powers and the Keisler order. In [Cha08] Chatzidakis observes that ω-free PAC fields have TP_2.

A brief description of the results in this paper.

In Section 1 we introduce inp-patterns, burden, establish some of their basic properties and demonstrate that burden is sub-multiplicative: that is, if $\mathrm{bdn}(a/C) < \kappa$ and $\mathrm{bdn}(b/aC) < \lambda$, then $\mathrm{bdn}(ab/C) < \kappa \times \lambda$. As an application we show that the value of the invariant of a theory $\kappa_{\mathrm{inp}}(T)$ does not depend on the number of variables used in the computation. This answers a question of Shelah from [She90] and shows in particular that if T has TP_2, then some formula $\phi(x, y)$ with x a singleton has TP_2.

In Section 2 we describe the place of NTP_2 in the classification hierarchy of first-order theories and the relationship of burden to dp-rank in NIP theories and to weight in simple theories. We also recall some combinatorial "structure / non-structure" dichotomy due to Shelah.

Section 3 is devoted to forking (and dividing) in NTP_2 theories. After discussing strictly invariant types, we give a characterization of NTP_2 in terms of the appropriate variants of Kim's lemma, local character and bounded weight relatively to strict non-forking. As an application we consider theories with dependent dividing (i.e. whenever $p \in S(N)$ divides over $M \prec N$, there some $\phi(x, a) \in p$ dividing over M and such that $\phi(x, y)$ is NIP) and show that any theory with dependent dividing is NTP_2. Finally we observe that the the analysis from [CK12] generalizes to a situation when one is working inside an NTP_2 type in an arbitrary theory.

A famous equation of Shelah "NIP = stability + dense linear order" turned out to be a powerful ideological principle, at least at the early stages of the development of NIP theories. In this paper the equation "NTP_2 = simplicity + NIP" plays an important role. In particular, it seems very natural to consider two extremal kinds of types in NTP_2 theories (and in general) — simple types and NIP types. While it is perfectly possible for an NTP_2 theory to have neither, they form important special cases and are not entirely understood.

In section 4 we look at NIP types. In particular we show that the results of the previous section on forking localized to a type combined with honest definitions from [CS10] allow to omit the global NTP_2 assumption in the theorem of [KS12], thus proving that dp-rank of a type in arbitrary theory is always witnessed by mutually indiscernible sequences of its realizations, after adding some parameters (see Theorem 51). We also observe that in an NTP_2 theory, a type is NIP if and only if every extension of it has only boundedly many global non-forking extensions.

In Section 5 we consider simple types (defined as those type for which every completion satisfies the local character), first in arbitrary theories and then in NTP_2. While it is more or less immediate that on the set of realizations of a simple type forking satisfies all the properties of forking in simple

theories, the interaction between the realizations of a simple type and arbitrary tuples seems more intricate. We establish full symmetry between realizations of a simple type and arbitrary elements, answering a question of Casanovas in the case of NTP_2 theories (showing that simple types are co-simple, see Definition 66). Then we show that simple types are characterized as those satisfying the co-independence theorem and that co-simple stably embedded types are simple (so in particular a theory is simple if and only if it is NTP_2 and satisfies the independence theorem).

Section 6 is devoted to examples. We give an Ax-Kochen-Ershov type statement: a Henselian valued field of characteristic $(0,0)$ is NTP_2 (strong, of finite burden) if and only if the residue field is NTP_2 (the residue field and the value group are strong, of finite burden respectively). This is parallel to the result of Delon for NIP [Del81], and generalizes a result of Shelah for strong dependence [She05]. It follows that the valued fields of Hahn series over pseudo-finite fields are NTP_2. In particular, the theory of the ultra-product of p-adics is NTP_2 (and in fact strong, of finite burden). We also show that expanding a geometric NTP_2 theory by a generic predicate (Chatzidakis-Pillay style [CP98]) preserves NTP_2.

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Preliminaries

As usual, we will be working in a monster model $\mathbb{M}$ of a complete first-order theory T. We will not be distinguishing between elements and tuples unless explicitly stated.

0.1. Mutually indiscernible sequences and arrays.

Definition 1. We will often be considering collections of sequences $(\bar{a}_\alpha)_{\alpha<\kappa}$ with $\bar{a}_\alpha = (a_{\alpha,i})_{i<\lambda}$ (where each $a_{\alpha,i}$ is a tuple, maybe infinite). We say that they are *mutually indiscernible* over a set C if $\bar{a}_\alpha$ is indiscernible over $C\bar{a}_{\neq\alpha}$ for all $i<\kappa$. We will say that they are *almost mutually indiscernible* over C if $\bar{a}_\alpha$ is indiscernible over $C\bar{a}_{<\alpha}\left(a_{\beta,0}\right)_{\beta>\alpha}$. Sometimes we call $(a_{\alpha,i})_{\alpha<\kappa,i<\lambda}$ an *array*. We say that $\left(\bar{b}_\alpha\right)_{\alpha<\kappa'}$ is a *sub-array* of $(\bar{a}_\alpha)_{\alpha<\kappa}$ if for each $\alpha<\kappa'$ there is $\beta_\alpha<\kappa$ such that $\bar{b}_\alpha$ is a sub-sequence of $\bar{a}_{\beta_\alpha}$. We say that an array is mutually indiscernible (almost mutually indiscernible) if rows are mutually indiscernible (resp. almost mutually indiscernible). Finally, an array is *very indiscernible* if it is mutually indiscernible and in addition the sequence of rows $(\bar{a}_\alpha)_{\alpha<\kappa}$ is an indiscernible sequence.

The following lemma follows easily by a repeated use of the usual "Erdös-Rado" and Ramsey theorems, and will be constantly used for finding indiscernible arrays.

Lemma 2. (1) *For any small set C and cardinal κ there is λ such that:*

If $A = (a_{\alpha,i})_{\alpha<n,i<\lambda}$ is an array, $n < \omega$ and $|a_{\alpha,i}| \leq \kappa$, then there is an array $B = (b_{\alpha,i})_{\alpha<n,i<\omega}$ with rows mutually indiscernible over C and such that every finite sub-array of B has the same type over C as some sub-array of A.

(2) *Let C be small set and $A = (a_{\alpha,i})_{\alpha<n,i<\omega}$ be an array with $n < \omega$. Then for any finite $\Delta \in L(C)$ and $N < \omega$ we can find Δ-mutually indiscernible sequences $\left(a_{\alpha,i_{\alpha,0}}, ..., a_{\alpha,i_{\alpha,N}}\right) \subset \bar{a}_\alpha$, $\alpha < n$.*

Lemma 3. *Let $(\bar{a}_\alpha)_{\alpha<\kappa}$ be almost mutually indiscernible over C. Then there are $(\bar{a}'_\alpha)_{\alpha<\kappa}$, mutually indiscernible over C and such that $\bar{a}'_\alpha \equiv_{a_{\alpha,0}} \bar{a}_\alpha$ for all $\alpha < \kappa$.*

Proof. By Lemma 2, taking an automorphism, and compactness. □

Definition 4. Given a set of formulas Δ, let $R(\kappa, \Delta)$ be the minimal length of a sequence sufficient for the existence of a Δ-indiscernible sub-sequence of length κ. For example, for finite Δ, $R(\kappa, \Delta) = \kappa$ for any infinite κ and and $R(n, \Delta)$ is finite for any $n \in \omega$.

Remark 5. Let $(\bar{a}_i)$ be a mutually indiscernible array over A. Then it is still a mutually indiscernible over $\operatorname{acl}(A)$.

0.2. **Invariant types.** We recall that

Fact 6. *(see e.g.* [HP11]*) Let $p(x)$ be a global type invariant over a set C (that is $\phi(x, a) \in p$ if and only if $\phi(x, \sigma(a)) \in p$ for any $\sigma \in \operatorname{Aut}(\mathbb{M}/C)$). For any set $D \supseteq C$, and an ordinal α, let the sequence $\bar{c} = \langle c_i \mid i < \alpha \rangle$ be such that $c_i \models p|_{Dc_{<i}}$. Then $\bar{c}$ is indiscernible over D and its type over D does not depend on the choice of $\bar{c}$. Call this type $p^{(\alpha)}|_D$, and let $p^{(\alpha)} = \bigcup_{D \supseteq C} p^{(\alpha)}|_D$. Then $p^{(\alpha)}$ also does not split over C.*

Finally, we assume some acquaintance with the basics of simple (e.g. [Cas07]) and NIP (e.g. [Adl08]) theories.

1. Burden and κ_{inp}

Let $p(x)$ be a (partial) type.

Definition 7. An inp-*pattern* in $p(x)$ of depth κ consists of $(a_{\alpha,i})_{\alpha<\kappa,i<\omega}$, $\phi_\alpha(x, y_\alpha)$ and $k_\alpha < \omega$ such that

- $\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ is k_α-inconsistent, for each $\alpha < \kappa$
- $\left\{\phi_\alpha(x, a_{\alpha,f(\alpha)})\right\}_{\alpha<\kappa} \cup p(x)$ is consistent, for any $f : \kappa \to \omega$.

The *burden* of $p(x)$, denoted $\operatorname{bdn}(p)$, is the supremum of the depths of all inp-patterns in $p(x)$. By $\operatorname{bdn}(a/C)$ we mean $\operatorname{bdn}(tp(a/C))$.

Obviously, $p(x) \subseteq q(x)$ implies $\mathrm{bdn}(p) \geq \mathrm{bdn}(q)$ and $\mathrm{bdn}(p) = 0$ if and only if p is algebraic. Also notice that $\mathrm{bdn}(p) < \infty \Leftrightarrow \mathrm{bdn}(p) < |T|^+$ by compactness.

First we observe that it is sufficient to look at mutually indiscernible inp-patterns.

Lemma 8. *For $p(x)$ a (partial) type over C, the following are equivalent:*

1. *There is an* inp-pattern *of depth κ in $p(x)$.*
2. *There is an array $(\bar{a}_\alpha)_{\alpha<\kappa}$ with rows mutually indiscernible over C and $\phi_\alpha(x, y_\alpha)$ for $\alpha < \kappa$ such that:*
   - *$\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ is inconsistent for every $\alpha < \kappa$*
   - *$p(x) \cup \{\phi_\alpha(x, a_{\alpha,0})\}_{\alpha<\kappa}$ is consistent.*
3. *There is an array $(\bar{a}_\alpha)_{\alpha<\kappa}$ with rows almost mutually indiscernible over C with the same properties.*

Proof. (1)⇒(2) is a standard argument using Lemma 2 and compactness, (2)⇒(3) is clear and (3)⇒(1) is an easy reverse induction plus compactness. □

We will need the following technical lemma.

Lemma 9. *Let $(\bar{a}_\alpha)_{\alpha<\kappa}$ be a mutually indiscernible array over C and b given. Let $p_\alpha(x, a_{\alpha,0}) =$ $\mathrm{tp}(b/a_{\alpha,0}C)$, and assume that $p^\infty(x) = \bigcup_{\alpha<\kappa, i<\omega} p_\alpha(x, a_{\alpha,i})$ is consistent. Then there are $(\bar{a}'_\alpha)_{\alpha<\kappa}$ such that:*

1. *$\bar{a}'_\alpha \equiv_{a_{\alpha,0}C} \bar{a}_\alpha$ for all $\alpha < \kappa$*
2. *$(\bar{a}'_\alpha)_{\alpha<\kappa}$ is a mutually indiscernible array over Cb.*

Proof. It is sufficient to find b' such that $b' \equiv_{a_{\alpha,0}C} b$ for all $\alpha < \kappa$ and $(\bar{a}_\alpha)_{\alpha<\kappa}$ is mutually indiscernible over $b'C$ (then applying an automorphism over C to conclude). Let $b^\infty \models p^\infty(x)$. By Lemma 2, for any finite $\Delta \in L(C)$, $S \subseteq \kappa$ and $n < \omega$, there is a $\Delta(b^\infty)$-mutually indiscernible sub-array $\left(a'_{\alpha,i}\right)_{\alpha\in S, i<n}$ of $(\bar{a}_\alpha)_{\alpha\in S}$. Let σ be an automorphism over C sending $\left(a'_{\alpha,i}\right)_{\alpha\in S, i<n}$ to $(a_{\alpha,i})_{\alpha\in S, i<n}$ and $b' = \sigma(b^\infty)$. Then $(a_{\alpha,i})_{\alpha\in S, i<n}$ is $\Delta(b')$-mutually indiscernible and $b' \models$ $\bigcup_{\alpha\in S} p_\alpha(x, a_{\alpha,0})$, so $b' \equiv_{a_{\alpha,0}C} b$. Conclude by compactness. □

Next lemma provides a useful equivalent way to compute the burden of a type.

Lemma 10. *The following are equivalent for a partial type $p(x)$ over C:*

1. *There is no* inp-pattern *of depth κ in p.*
2. *For any $b \models p(x)$ and $(\bar{a}_\alpha)_{\alpha<\kappa}$, an almost mutually indiscernible array over C, there is $\beta < \kappa$ and $\bar{a}'$ indiscernible over bC and such that $\bar{a}' \equiv_{a_{\beta,0}C} \bar{a}_\beta$.*
3. *For any $b \models p(x)$ and $(\bar{a}_\alpha)_{\alpha<\kappa}$, a mutually indiscernible array over C, there is $\beta < \kappa$ and $\bar{a}'$ indiscernible over bC and such that $\bar{a}' \equiv_{a_{\beta,0}C} \bar{a}_\beta$.*

Proof. (1)⇒(2): So let $(\bar{a}_\alpha)_{\alpha<\kappa}$ be almost mutually indiscernible over C and $b \models p(x)$ given. Let $p_\alpha(x, a_{\alpha,0}) = \operatorname{tp}(b/a_{\alpha,0}C)$ and let $p_\alpha(x) = \bigcup_{i<\omega} p_\alpha(x, a_{\alpha,i})$.

Assume that p_α is inconsistent for each α, by compactness and indiscernibility of $\bar{a}_\alpha$ over C there is some $\phi_\alpha(x, a_{\alpha,0}c_\alpha) \in p_\alpha(x, a_{\alpha,0})$ with $c_\alpha \in C$ such that $\{\phi_\alpha(x, a_{\alpha,i}c_\alpha)\}_{i<\omega}$ is k_α-inconsistent. As $b \models \{\phi_\alpha(x, a_{\alpha,0}c_\alpha)\}_{\alpha<\kappa}$, by almost indiscernibility of $(\bar{a}_\alpha)_{\alpha<\kappa}$ over C and Lemma 8 we find an inp-pattern of depth κ in p – a contradiction.

Thus $p_\beta(x)$ is consistent for some $\beta < \kappa$. Then we can find $\bar{a}'$ which is indiscernible over bC and such that $\bar{a}' \equiv_{a_{\beta,0}C} \bar{a}_\beta$ by Lemma 9.

(2)⇒(3) is clear.

(3)⇒(1): Assume that there is an inp-pattern of depth κ in $p(x)$. By Lemma 8 there is an inp-pattern $(\bar{a}_\alpha, \phi_\alpha, k_\alpha)_{\alpha<\kappa}$ in $p(x)$ with $(\bar{a}_\alpha)_{\alpha<\kappa}$ a mutually indiscernible array over C. Let $b \models p(x) \cup \{\phi_\alpha(x, a_{\alpha,0})\}_{\alpha<\kappa}$. On the one hand $\models \phi_\alpha(b, a_{\alpha,0})$, while on the other $\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ is inconsistent, thus it is impossible to find an $\bar{a}'_\alpha$ as required for any $\alpha < \kappa$. □

Theorem 11. *If there is an* inp*-pattern of depth* $\kappa_1 \times \kappa_2$ *in* $\operatorname{tp}(b_1b_2/C)$*, then either there is an* inp*-pattern of depth* κ_1 *in* $\operatorname{tp}(b_1/C)$ *or there is an* inp*-pattern of depth* κ_2 *in* $\operatorname{tp}(b_2/b_1C)$*.*

Proof. Assume not. Without loss of generality $C = \emptyset$, and let $(\bar{a}_\alpha)_{\alpha\in\kappa_1\times\kappa_2}$ be a mutually indiscernible array. By induction on $\alpha < \kappa_1$ we choose $\bar{a}'_\alpha$ and $\beta_\alpha \in \kappa_2$ such that:

1. $\bar{a}'_\alpha$ is indiscernible over $b_2\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}$.
2. $\operatorname{tp}(\bar{a}'_\alpha/a_{(\alpha,\beta_\alpha),0}\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}) = \operatorname{tp}(\bar{a}_{(\alpha,\beta_\alpha)}/a_{(\alpha,\beta_\alpha),0}\bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)})$.
3. $\bar{a}'_{\leq\alpha} \cup \bar{a}_{\geq(\alpha+1,0)}$ is a mutually indiscernible array.

For $\alpha = -1$, (1) and (2) are empty conditions and (3) is the assumption. Now assume we have managed up to α, and we need to choose $\bar{a}'_\alpha$ and β_α. Let $D = \bar{a}'_{<\alpha}\bar{a}_{\geq(\alpha+1,0)}$. As $\left(\bar{a}_{(\alpha,\delta)}\right)_{\delta\in\kappa_2}$ is a mutually indiscernible array over D by (3) and there is no inp-pattern of depth κ_2 in $\operatorname{tp}(b_2/D)$, by Lemma 10(3) there is some $\beta_\alpha < \kappa_2$ and $\bar{a}'_\alpha$ indiscernible over b_2D (which gives us (1)) and such that $\operatorname{tp}(\bar{a}'_\alpha/a_{(\alpha,\beta_\alpha),0}D) = \operatorname{tp}(\bar{a}_{(\alpha,\beta_\alpha)}/a_{(\alpha,\beta_\alpha),0}D)$ (which together with the inductive assumption gives us (2) and (3)).

So we have carried out the induction. Now it is easy to see by (1), noticing that the first elements of $\bar{a}'_\alpha$ and $\bar{a}_{(\alpha,\beta_\alpha)}$ are the same by (2), that $(\bar{a}'_\alpha)_{\alpha<\kappa_1}$ is an almost mutually indiscernible array over b_2. By Lemma 3, we may assume that in fact $(\bar{a}'_\alpha)_{\alpha<\kappa_1}$ is a mutually indiscernible array over b_2.

As there is no inp-pattern of depth κ_1 in $\operatorname{tp}(b_1/b_2)$, by Lemma 10 there is some $\gamma < \kappa_1$ and $\bar{a}$ indiscernible over b_1b_2 and such that $\bar{a} \equiv_{a'_{\gamma,0}} \bar{a}'_\gamma \equiv_{a_{(\gamma,\beta_\gamma),0}} \bar{a}_{(\gamma,\beta_\gamma)}$. As $(\bar{a}_\alpha)_{\alpha\in\kappa_1\times\kappa_2}$ was arbitrary, by Lemma 10(3) this implies that there is no inp-pattern of depth $\kappa_1 \times \kappa_2$ in $\operatorname{tp}(b_1b_2)$. □

Corollary 12. *"Sub-multiplicativity" of burden: If* $\operatorname{bdn}(a_i) < k_i$ *for* $i < n$ *with* $k_i \in \omega$*, then* $\operatorname{bdn}(a_0...a_{n-1}) < \prod_{i<n} k_i$*.*

We note that in the case of NIP theories it is known that burden is not only sub-multiplicative, but actually sub-additive [KOU11]. See [BYC] concerning the question of sub-additivity of burden in NTP_2 theories.

Definition 13. For $n<\omega$, we let $\kappa^n_{\mathrm{inp}(T)}$ be the first cardinal κ such that there is no inp-pattern $(\bar{a}_\alpha,\phi_\alpha(x,y_\alpha),k_\alpha)$ of depth κ with $|x|\leq n$. And let $\kappa_{\mathrm{inp}}(T)=\sup_{n<\omega}\kappa^n_{\mathrm{inp}}(T)$. Notice that $\kappa^m_{\mathrm{inp}}\geq\kappa^n_{\mathrm{inp}}(T)\geq n$ for all $n<m$, just because of having the equality in the language, and thus $\kappa_{\mathrm{inp}(T)}\geq\aleph_0$.

We can use the previous theorem to answer a question of Shelah [She90, Ch. III, Question 7.5].

Corollary 14. *$\kappa_{\mathrm{inp}}(T)=\kappa^n_{\mathrm{inp}}(T)=\kappa^1_{\mathrm{inp}}(T)$, as long as κ^n_{inp} is infinite for some $n<\omega$.*

2. NTP_2 and its place in the classification hierarchy

The aim of this section is to (finally) define NTP_2, describe its place in the classification hierarchy of first-order theories and what burden amounts to in the more familiar situations.

Definition 15. A formula $\phi(x,y)$ has TP_2 if there is an array $(a_{\alpha,i})_{\alpha,i<\omega}$ such that $\{\phi(x,a_{\alpha,i})\}_{i<\omega}$ is 2-inconsistent for every $\alpha<\omega$ and $\left\{\phi(x,a_{\alpha,f(\alpha)})\right\}_{\alpha<\omega}$ is consistent for any $f:\omega\to\omega$. Otherwise we say that $\phi(x,y)$ is NTP_2, and T is NTP_2 if every formula is.

Lemma 16. *The following are equivalent for T:*

(1) *Every formula $\phi(x,y)$ with $|x|\leq n$ is NTP_2.*
(2) $\kappa^n_{\mathrm{inp}}(T)\leq|T|^+$.
(3) $\kappa^n_{\mathrm{inp}}(T)<\infty$.
(4) $\mathrm{bdn}(b/C)<|T|^+$ *for all b and C, with $|b|=n$.*

Proof. (1)$\Rightarrow$(2): Assume we have a mutually indiscernible inp-pattern $(\bar{a}_\alpha,\phi_\alpha(x,y_\alpha),k_\alpha)_{\alpha<|T|^+}$ of depth $|T|^+$. By pigeon-hole we may assume that $\phi_\alpha(x,y_\alpha)=\phi(x,y)$ and $k_\alpha=k$. Then by Ramsey and compactness we may assume in addition that $(\bar{a}_\alpha)$ is a very indiscernible array. If $\{\phi(x,a_{\alpha,0})\wedge\phi(x,a_{\alpha,1})\}_{\alpha<n}$ is inconsistent for some $n<\omega$, then taking $b_{\alpha,i}=a_{n\alpha,i}a_{n\alpha+1,i}...a_{n\alpha+n-1,i}$, $\left(\bigwedge_{i<n}\phi(x,y_i),\bar{b}_\alpha,2\right)_{\alpha<\omega}$ is an inp-pattern. Otherwise $\{\phi(x,a_{\alpha,0})\wedge\phi(x,a_{\alpha,1})\}_{\alpha<\omega}$ is consistent, then taking $b_{\alpha,i}=a_{\alpha,2i}a_{\alpha,2i+1}$ we conclude that $\left(\phi(x,y_1)\wedge\phi(x,y_2),\bar{b}_\alpha,\left[\frac{k}{2}\right]\right)_{\alpha<\omega}$ is an inp-pattern. Repeat if necessary.

The other implications are clear by compactness. ☐

Remark 17. (1) implies (2) is from [Adl07].

It follows from the lemma and Theorem 14 that if T has TP_2, then some formula $\phi(x,y)$ with $|x|=1$ has TP_2. From Lemma 86 it follows that if $\phi_1(x,y_1)$ and $\phi_2(x,y_2)$ are NTP_2, then $\phi_1(x,y_1)\vee\phi_2(x,y_2)$ is NTP_2. This, however, is the only Boolean operation preserving NTP_2.

Definition 18. [Adler] T is called *strong* if there is no inp-pattern of infinite depth in it. It is clearly a subclass of NTP_2 theories.

Proposition 19. *If $\phi(x,y)$ is* NIP*, then it is* NTP_2*.*

Proof. Let $(a_{\alpha,j})_{\alpha,j<\omega}$ be an array witnessing that $\phi(x,y)$ has TP_2. But then for any $s\subseteq\omega$, let $f(\alpha)=0$ if $\alpha\in s$, and $f(\alpha)=1$ otherwise. Let $d\models\left\{\phi(x,a_{\alpha,f(\alpha)})\right\}$. It follows that $\phi(d,a_{\alpha,0})\Leftrightarrow\alpha\in s$. $\square$

We recall the definition of dp-rank (e.g. [KOU11]):

Definition 20. We let the dp-rank of p, denoted $\mathrm{dprk}(p)$, be the supremum of κ for which there are $b\models p$ and mutually indiscernible over C (a set containing the domain of p) sequences $(\bar{a}_\alpha)_{\alpha<\kappa}$ such that none of them is indiscernible over bC.

Fact 21. *The following are equivalent for a partial type $p\left(x\right)$ (by Ramsey and compactness):*

1. $\mathrm{dprk}\left(p\right)>\kappa$.
2. *There is an ict-pattern of depth κ in $p\left(x\right)$, that is $\left(\bar{a}_i,\varphi_i\left(x,y_i\right),k_i\right)_{i<\kappa}$ such that $p\left(x\right)\cup\left\{\varphi_i\left(x,a_{i,s(i)}\right)\right\}_{i<\kappa}\cup\left\{\neg\varphi_i\left(x,a_{i,j}\right)\right\}_{s(i)\neq j<\kappa}$ is consistent for every $s:\kappa\to\omega$.*

It is easy to see that every inp-pattern with mutually indiscernible rows gives an ict-pattern of the same depth. On the other hand, if T is NIP then every ict-pattern gives an inp-pattern of the same depth (see [Adl07, Section 3]). Thus we have:

Fact 22.
1. *For a partial type $p(x)$,* $\mathrm{bdn}(p)\geq\mathrm{dprk}(p)$*. And if $p(x)$ is an* NIP *type, then* $\mathrm{bdn}(p)=\mathrm{dprk}(p)$
2. *T is strongly dependent $\Leftrightarrow$ T is* NIP *and strong.*

Proposition 23. *If T is simple, then it is* NTP_2*.*

Proof. Of course, inp-pattern of the form $\left(\bar{a}_\alpha,\phi(x,y),k\right)_{\alpha<\omega}$ witnesses the tree property. $\square$

Moreover,

Fact 24. [Adl07, Proposition 8] *Let T be simple. Then the burden of a partial type is the supremum of the weights of its complete extensions. And T is strong if and only if every type has finite burden.*

Definition 25. [Shelah] $\phi(x,y)$ is said to have TP_1 if there are $(a_\eta)_{\eta\in\omega^{<\omega}}$ and $k\in\omega$ such that:

- $\left\{\phi(x,a_{\eta|i})\right\}_{i\in\omega}$ is consistent for any $\eta\in\omega^\omega$
- $\{\phi(x,a_{\eta_i})\}_{i<k}$ is inconsistent for any mutually incomparable $\eta_0,...,\eta_{k-1}\in\omega^{<\omega}$.

Fact 26. [She90, III.7.7, III.7.11] *Let T be* NTP$_2$*, $q(y)$ a partial type and $\phi(x,y)$ has* TP *witnessed by $(a_\eta)_{\eta\in\omega^{<\omega}}$ with $a_\eta \models q$, and such that in addition $\left\{\phi(x,a_{\eta|i})\right\}_{i\in\omega} \cup p(x)$ is consistent for any $\eta \in \omega^\omega$. Then some formula $\psi(x,\bar{y}) = \bigwedge_{i<k}\phi(x,y_i)\wedge\chi(x)$ (where $\chi(x)\in p(x)$) has* TP$_1$*, witnessed by (b_η) with $b_\eta \subseteq q(\mathbb{M})$ and such that $\left\{\phi(x,b_{\eta|i})\right\}_{i\in\omega}\cup p(x)$ is consistent.*

It is not stated in exactly the same form there, but immediately follows from the proof. See [Adl07, Section 4] and [KKS12, Theorem 6.6] for a more detailed account of the argument. See [KK11] for more details on NTP$_1$.

Example 27.
1. Triangle free random graph (i.e. the model companion of the theory of graphs without triangles) has TP$_2$.
2. The theories of free roots of the random graph (as defined and studied in [CW04]) have TP$_2$. In particular, the rational Urysohn space has TP$_2$.

Proof. (1): We can find $(a_{ij}b_{ij})_{ij<\omega}$ such that $R(a_{ij},b_{ik})$ for every i and $j\neq k$, and this are the only edges around. But then $\{xRa_{ij}\wedge xRb_{ij}\}_{j<\omega}$ is 2-inconsistent for every i as otherwise it would have created a triangle, while $\left\{xRa_{if(i)}\wedge xRb_{if(i)}\right\}_{i<\omega}$ is consistent for any $f:\omega\to\omega$.

(2): Let $(a_{i,j})_{i,j<\omega}$ be such that $d(a_{i,j},a_{i,j'})=3$ for all $i,j\neq j'<\omega$ and $d(a_{i,j},a_{i',j'})=2$ for all $i\neq i',j,j'<\omega$ - possible to find by model completeness as the triangular inequality is not violated. But then $\{xR_1a_{i,j}\}_{j<\omega}$ is inconsistent for every i, while $\left\{xR_1a_{i,f(i)}\right\}_{i<\omega}$ is consistent for any $f:\omega\to\omega$. $\square$

In fact it is known that the triangle-free random graph is rosy and 2-dependent (in the sense of [She07]), thus there is no implication between rosiness and NTP$_2$, and between k-dependence and NTP$_2$ for $k>1$. We also remark that in [She90, Exercise III.7.12] Shelah suggests an example of a theory satisfying NTP$_2$ + NSOP which is not simple.

3. Forking in NTP$_2$

In [Kim01, Theorem 2.4] Kim gives several equivalents to the simplicity of a theory in terms of the behavior of forking and dividing.

Fact 28. *The following are equivalent:*
1. *T is simple.*
2. *$\phi(x,a)$ divides over A if and only if $\{\phi(x,a_i)\}_{i<\omega}$ is inconsistent for every Morley sequence $(a_i)_{i<\omega}$ over A.*
3. *Dividing in T satisfies local character.*

In this section we show an analogous characterization of NTP$_2$. But first we recall some facts about forking and dividing in NTP$_2$ theories and introduce some terminology.

Definition 29. (1) A type $p(x) \in S(C)$ is *strictly invariant* over A if it is Lascar invariant over A and for any small $B \subseteq C$ and $a \models p|_B$, we have that $\mathrm{tp}(B/aA)$ does not divide over A (we can replace "does not divide" by "does not fork" $C = \mathbb{M}$). For example, a definable type or a global type which is both an heir and a coheir over M, are strictly invariant over M.

(2) We will write $a \mathop{\smile\!\!\!\!\!\!|}\,_c^{\mathrm{ist}} b$ when $\mathrm{tp}(a/bc)$ can be extended to a global type $p(x)$ strictly invariant over A.

(3) We say that $(a_i)_{<\omega}$ is a strict Morley sequence over A if it is indiscernible over A and $a_i \mathop{\smile\!\!\!\!\!\!|}\,_A^{\mathrm{ist}} a_{<i}$ for all $i < \omega$.

(4) As usual, we will write $a \mathop{\smile\!\!\!\!\!\!|}\,_c^{u} b$ if $\mathrm{tp}(a/bc)$ is finitely satisfiable in c, $a \mathop{\smile\!\!\!\!\!\!|}\,_c^{d} b$ ($a \mathop{\smile\!\!\!\!\!\!|}\,_c^{f} b$) if $\mathrm{tp}(a/bc)$ does not divide (resp. does not fork) over c.

(5) We write $a \mathop{\smile\!\!\!\!\!\!|}\,_c^{i} b$ if $\mathrm{tp}(a/bc)$ can be extended to a global type $p(x)$ Lascar invariant over c. We point out that if $a \mathop{\smile\!\!\!\!\!\!|}\,_c^{i} b$ and $(b_i)_{i<\omega}$ is a c-indiscernible sequence with $b_0 = b$, then it is actually indiscernible over a.

(6) If T is simple, then $\mathop{\smile\!\!\!\!\!\!|}\,^{i} = \mathop{\smile\!\!\!\!\!\!|}\,^{\mathrm{ist}}$. And if T is NIP, then $\mathop{\smile\!\!\!\!\!\!|}\,^{i} = \mathop{\smile\!\!\!\!\!\!|}\,^{f}$.

(7) We say that a set A is an *extension base* if every type over A has a global non-forking extension. Every model is an extension base (because every type has a global coheir). A theory in which every set is an extension base is called extensible.

Strictly invariant types exist in any theory (but it is not true that every type over a model has a global extension which is strictly invariant over the same model). In fact, there are theories in which over any set there is some type without a global strictly invariant extension (see [CKS12]).

Lemma 30. *Let $p(x)$ be a global type invariant over A, and let $M \supset A$ be $|A|^+$-saturated. Then p is strictly invariant over M.*

Proof. It is enough to show that p is an heir over M. Let $\phi(x,c) \in p$. By saturation of M, $\mathrm{tp}(c/A)$ is realized by some $c' \in M$. But as p is invariant over A, $\phi(x,c') \in p$ as wanted. □

One of the main uses of strict invariance is the following criterion for making indiscernible sequences mutually indiscernible without changing their type over the first elements.

Lemma 31. *Let $(\bar{a}_i)_{i<\kappa}$ and C be given, with $\bar{a}_i$ indiscernible over C and starting with a_i. If $a_i \mathop{\smile\!\!\!\!\!\!|}\,_C^{\mathrm{ist}} a_{<i}$, then there are mutually C-indiscernible $\left(\bar{b}_i\right)_{i<\kappa}$ such that $\bar{b}_i \equiv_{a_i C} \bar{a}_i$.*

Proof. (1): Enough to show for finite κ by compactness. So assume we have chosen $\bar{a}'_0, ..., \bar{a}'_{n-1}$, and lets choose $\bar{a}'_n$. As $a_n \mathop{\smile\!\!\!\!\!\!|}\,_C^{\mathrm{ist}} a_{<n}$, there are $\bar{a}''_0 ... \bar{a}''_{n-1} \equiv_{C a_0 ... a_{n-1}} \bar{a}'_0 ... \bar{a}'_{n-1}$ and such that $a_n \mathop{\smile\!\!\!\!\!\!|}\,_C^{\mathrm{ist}} \bar{a}''_{<n}$. As $a_n \mathop{\smile\!\!\!\!\!\!|}\,_{C \bar{a}''_{<n, \neq j}}^{i} \bar{a}''_j$ for $j < n$, it follows by the inductive assumption and Definition 29(5) that $\bar{a}''_j$ is indiscernible over $a_n \bar{a}''_{\neq j}$. On the other hand $\bar{a}''_0 ... \bar{a}''_{n-1} \mathop{\smile\!\!\!\!\!\!|}\,_C^{f} a_n$, and so by basic

properties of forking there is some $\bar{a}_n' \equiv_{Ca_n} \bar{a}_n$ indiscernible over $\bar{a}_0'', ..., \bar{a}_{n-1}''$. Conclude by Lemma 3. $\square$

Remark 32. This argument is essentially from [She09, Section 5].

We recall a result about forking and dividing in NTP_2 theories from [CK12].

Fact 33. [CK12] *Let T be NTP_2 and $M \models T$.*

1. *Every $p \in S(M)$ has a global strictly invariant extension.*
2. *For any a, $\phi(x, a)$ divides over M if and only if $\phi(x, a)$ forks over M, if and only if for every $(a_i)_{i<\omega}$, a strict Morley sequence in $\mathrm{tp}(a/M)$, $\{\phi(x, a_i)\}_{i<\omega}$ is inconsistent.*
3. *In fact, just assuming that A is an extension base, we still have that $\phi(x, a)$ does not divide over A if and only if $\phi(x, a)$ does not fork over A.*

3.1. **Characterization of** NTP_2**.** Now we can give a method for computing the burden of a type in terms of dividing with each member of an $\mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}$-independent sequence.

Lemma 34. *Let $p(x)$ be a partial type over C. The following are equivalent:*

1. *There is an* inp*-pattern of depth κ in $p(x)$.*
2. *There is $d \models p(x)$, $D \supseteq C$ and $(a_\alpha)_{\alpha<\kappa}$ such that $a_\alpha \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}_D a_{<\alpha}$ and $d \not\mathop{\smile\!\!\!\!\!\mid}^{d}_D a_\alpha$ for all $\alpha < \kappa$.*

Proof. (1)⇒(2): Let $(\bar{a}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha)_{\alpha<\kappa}$ be an inp-pattern in $p(x)$ with $(\bar{a}_\alpha)$ mutually indiscernible over C. Let $q_\alpha(\bar{y}_\alpha)$ be a non-algebraic type finitely satisfiable in $\bar{a}_\alpha$ and extending $\mathrm{tp}(a_{\alpha 0}/C)$. Let $M \supseteq C\,(\bar{a}_\alpha)_{\alpha<\kappa}$ be $(|C| + \kappa)^+$-saturated. Then q_α is strictly invariant over M by Lemma 30. For $\alpha, i < \kappa$ let $b_{\alpha,i} \models q_\alpha \restriction_{M (b_{\alpha,j})_{\alpha<\kappa, j<i} (b_{\beta,i})_{\beta<\alpha}}$. Let $e_\alpha = b_{\alpha,\alpha}$. Now we have:

- $e_\alpha \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}_M e_{<\alpha}$: as $e_\alpha \models q_\alpha \restriction_{e_{<\alpha} M}$.
- there is $d \models p(x) \cup \{\phi_\alpha(x, e_\alpha)\}_{\alpha<\kappa}$: it is easy to see by construction that for any $\Delta \in L(C)$ and $\alpha_0 < ... < \alpha_{n-1} < \kappa$, if $\models \Delta(e_{\alpha_0}, ..., e_{\alpha_{n-1}})$, then $\models \Delta(a_{\alpha_0,i_0}, ..., a_{\alpha_{n-1},i_{n-1}})$ for some $i_0, ..., i_{n-1} < \omega$. By assumption on $(\bar{a}_\alpha)_{\alpha<\kappa}$ and compactness it follows that $p(x) \cup \{\phi_\alpha(x, e_\alpha)\}_{\alpha<\kappa}$ is consistent.
- $\phi_\alpha(x, e_\alpha)$ divides over M: notice that $(b_{\alpha,\alpha+i})_{i<\omega}$ is an M-indiscernible sequence starting with e_α, as $b_{\alpha,\alpha+i} \models q_\alpha \restriction_{M (b_{\alpha,\alpha+j})_{j<i}}$ and q_α is finitely satisfiable in M. As $\mathrm{tp}(\bar{b}_\alpha)$ is finitely satisfiable in $\bar{a}_\alpha$, we conclude that $\{\phi_\alpha(x, b_{\alpha,\alpha+i})\}_{i<\omega}$ is k_α-inconsistent.

(2)⇒(1): Let $d \models p(x)$, $D \supseteq C$ and $(a_\alpha)_{\alpha<\kappa}$ such that $a_\alpha \mathop{\smile\!\!\!\!\!\mid}^{\mathrm{ist}}_D a_{<\alpha}$ and $d \not\mathop{\smile\!\!\!\!\!\mid}^{f}_D a_\alpha$ for all $\alpha < \kappa$ be given. Let $\phi_\alpha(x, a_\alpha) \in \mathrm{tp}(d/a_\alpha D)$ be a formula dividing over D, and let $\bar{a}_\alpha$ indiscernible over D and starting with a_α witness it. By Lemma 8 we can find a $(\bar{a}_\alpha')_{\alpha<\kappa}$, mutually indiscernible over D and such that $\bar{a}_\alpha' \equiv_{a_\alpha D} \bar{a}_\alpha$. It follows that $\{\phi_\alpha(x, y_\alpha), \bar{a}_\alpha'\}_{\alpha<\kappa}$ is an inp-pattern of depth κ in $p(x)$. $\square$

Definition 35. We say that dividing satisfies *generic local character* if for every $A \subseteq B$ and $p(x) \in S(B)$ there is some $A' \subseteq B$ with $|A'| \leq |T|^+$ and such that: for any $\phi(x,b) \in p$, if $b \perp^{\mathrm{ist}}_A A'$, then $\phi(x,b)$ does not divide over AA'.

Of course, the local character of dividing implies the generic local character. We are ready to prove the main theorem of this section.

Theorem 36. *The following are equivalent:*

1. T *is* NTP_2.
2. T *has absolutely bounded* $\perp^{\mathrm{ist}}$*-weight: for every* M*,* b *and* $(a_i)_{i<|T|^+}$ *with* $a_i \perp^{\mathrm{ist}}_M a_{<i}$*,* $b \perp^d_M a_i$ *for some* $i < |T|^+$*.*
3. T *has bounded* $\perp^{\mathrm{ist}}$*-weight: for every* M *there is some* κ_M *such that given* b *and* $(a_i)_{i<\kappa_M}$ *with* $a_i \perp^{\mathrm{ist}}_M a_{<i}$*,* $b \perp^d_M a_i$ *for some* $i < \kappa_M$*.*
4. T *satisfies "Kim's lemma": for any* $M \models T$*,* $\phi(x,a)$ *divides over* M *if and only if* $\{\phi(x,a_i)\}_{i<\omega}$ *is inconsistent for every strict Morley sequence over* M*.*
5. *Dividing in* T *satisfies generic local character.*

Proof. (1) implies (2): Assume that there are M, b and $(a_i)_{i<|T|^+}$ with $a_i \perp^{\mathrm{ist}}_M a_{<i}$ and $b \not\perp^d_M a_i$ for all i. But then by Lemma 34 $\mathrm{bdn}(b/M) \geq |T|^+$, thus T has TP_2 by Lemma 16.

(2) implies (3) is clear.

(1) implies (4): by Fact 33(1)+(2).

(4) implies (3): assume that we have M, b and $(a_i)_{i<\kappa}$ such that, letting $\kappa = \beth_{\left(2^{|M|}\right)^+}$, $a_i \perp^{\mathrm{ist}}_M a_{<i}$ and $b \not\perp^d_M a_i$ for all $i < \kappa$. We may assume that dividing is always witnessed by the same formula $\phi(x,y)$. Extracting an M-indiscernible sequence $(a'_i)_{i<\omega}$ from $(a_i)_{i<\kappa}$ by Erdös-Rado, we get a contradiction to (4) as $\{\phi(x,a'_i)\}_{i<\omega}$ is still consistent, (a'_i) is a strict Morley sequence over M and $\phi(x,a'_0)$ divides over M.

(3) implies (1): Assume that $\varphi(x,y)$ has TP_2, let $A = (\bar{a}_\alpha)_{\alpha<\omega}$ with $\bar{a}_\alpha = (a_{\alpha i})_{i<\omega}$ be a very indiscernible array witnessing it (so rows are mutually indiscernible and the sequence of rows is indiscernible). Let $M \supset A$ be some $|A|^+$-saturated model, and assume that κ_M is as required by (3). Let $\lambda = \beth_{\left(2^{|M|}\right)^+}$ and $\mu = \left(2^{2^\lambda}\right)^+$. Adding new elements and rows by compactness, extend our very indiscernible array to one of the form $(\bar{a}_\alpha)_{\alpha\in\omega+\mu^*}$ with $\bar{a}_\alpha = (a_{\alpha i})_{i\in\lambda}$. By all the indiscernibility around it follows that $\bar{a}_\alpha \perp^u_A \bar{a}_{<\alpha}$ for all $\alpha < \mu$. As there can be at most 2^{2^λ} global types from $S_\lambda(\mathbb{M})$ that are finitely satisfiable in A, without loss of generality there is some $p(\bar{x}) \in S_\lambda(\mathbb{M})$ finitely satisfiable in A and such that $\bar{a}_\alpha \models p(\bar{x})|_{A\bar{a}_{<\alpha}}$.

By Lemma 30, $p(\bar{x})$ is strictly invariant over M. We choose $\left(\bar{b}_\alpha\right)_{\alpha<\kappa_M}$ such that $\bar{b}_\alpha \models p|_{M\bar{b}_{<\alpha}}$.

By the choice of λ and Erdös-Rado, for each $\alpha < \kappa_M$ there is $i_\alpha < \lambda$ and $\bar{d}_\alpha$ such that $\bar{d}_\alpha$ is an M-indiscernible sequence starting with $b_{\alpha i_\alpha}$ and such that type of every finite subsequence of it is realized by some subsequence of $\bar{b}_\alpha$. Now we have:

- $d_{\alpha 0} \mathop{\smile\!\!\!\!\!\!|}\,^{\text{ist}}_{M} d_{<\alpha 0}$ (as $d_{\alpha 0} = b_{\alpha i_\alpha}$ and $\bar{b}_\alpha \mathop{\smile\!\!\!\!\!\!|}\,^{\text{ist}}_{M} \bar{b}_{<\alpha}$),
- $\varphi(x, d_{\alpha 0})$ divides over M (as $\bar{d}_\alpha$ is M-indiscernible and $\{\varphi(x, d_{\alpha i})\}_{i\in\omega}$ is inconsistent by construction),
- $\{\varphi(x, d_{\alpha 0})\}_{\alpha<\kappa_M}$ is consistent (follows by construction).

Taking some $c \models \{\varphi(x, d_{\alpha 0})\}_{\alpha<\kappa_M}$ we get a contradiction to (3).

(5) implies (2): Let $p(x) = \operatorname{tp}(b/B)$ with $B = M \cup \bigcup_{i<|T|^+} a_i$. Letting $A = M$, it follows by generic local character that there is some $A' \subseteq B$ with $|A'| \leq |T|$, such that $b \mathop{\smile\!\!\!\!\!\!|}\,^{d}_{MA'} a$ for any $a \in B$ with $a \mathop{\smile\!\!\!\!\!\!|}\,_{M} A'$. Let $i \in |T|$ be such that $i > \{j : a_j \in A'\}$. Then $a_i \mathop{\smile\!\!\!\!\!\!|}\,^{\text{ist}}_{M} A$, but also $b \not\mathop{\smile\!\!\!\!\!\!|}\,^{d}_{MA'} a_i$ (by left transitivity as $A' \mathop{\smile\!\!\!\!\!\!|}\,^{d}_{M} a_i$ and $b \not\mathop{\smile\!\!\!\!\!\!|}\,^{d}_{M} a_i$) — a contradiction.

(1) implies (5): Let $p(x) \in S(B)$ and $A \subseteq B$ be given. By induction on $i < |T|^+$ we try to choose $a_i \in B$ and $\varphi_i(x, a_i) \in p$ such that $a_i \mathop{\smile\!\!\!\!\!\!|}\,^{\text{ist}}_{A} a_{<i}$ and $\varphi_i(x, a_i)$ divides over $a_{<i}A$. But then by Lemma 34 $\operatorname{bdn}(b/A) \geq |T|^+$, thus T has TP_2 by Lemma 16. So we had to get stuck, and letting $A' = \bigcup a_i$ witnesses the generic local character. $\square$

Remark 37. (1) The proof of the equivalences shows that in (2) and (3) we may replace $a \mathop{\smile\!\!\!\!\!\!|}\,^{\text{ist}}_{C} b$ by "$\operatorname{tp}(a/bC)$ extends to a global type which is both an heir and a coheir over C".

(2) From the proof one immediately gets a similar characterization of strongness. Namely, the following are equivalent:
   (a) T is strong.
   (b) For every M, finite (or even singleton) b and $(a_i)_{i<\omega}$ with $a_i \mathop{\smile\!\!\!\!\!\!|}\,^{\text{ist}}_{M} a_{<i}$, $b \mathop{\smile\!\!\!\!\!\!|}\,^{d}_{M} a_i$ for some $i < \omega$.
   (c) For every $A \subseteq B$ and $p(x) \in S(B)$ there is some *finite* $A' \subseteq B$ such that: for any $\phi(x, b) \in p$, if $b \mathop{\smile\!\!\!\!\!\!|}\,^{\text{ist}}_{A} A'$, then $\phi(x, b)$ does not divide over AA'.

If we are working over a somewhat saturated model and consider only small sets, then we actually have the generic local character with respect to $\mathop{\smile\!\!\!\!\!\!|}\,^{u}$ in the place of $\mathop{\smile\!\!\!\!\!\!|}\,^{\text{ist}}$.

Lemma 38. *Let $(\bar{a}_i)_{i<\kappa}$ and C be given, $\bar{a}_i$ starting with a_i. If $\bar{a}_i$ is indiscernible over $\bar{a}_{<i}C$ and $a_i \mathop{\smile\!\!\!\!\!\!|}\,^{i}_{C} a_{<i}$, then $(\bar{a}_i)_{i<\kappa}$ is almost mutually indiscernible over C.*

Proposition 39. *Let T be NTP_2. Let M be κ-saturated, $p(x) \in S(M)$ and $A \subset M$ of size $< \kappa$. Then there is $A \subseteq A' \subset M$ of size $< \kappa$ such that for any $\phi(x, a) \in p$, if $a \mathop{\smile\!\!\!\!\!\!|}\,^{i}_{A} A'$ then $\phi(x, a)$ does not fork over A'.*

Proof. Assume not, then we can choose inductively on $\alpha < |T|^+$:

(1) $\bar{a}_\alpha \subseteq M$ such that $a_{\alpha,0} \mathop{\smile\!\!\!\!\!\!|}\,^{i}_{A} A_\alpha$ and $\bar{a}_\alpha$ is A_α-indiscernible, $A_\alpha = A \cup \bigcup_{\beta<\alpha} \bar{a}_\beta$.
(2) $\phi_\alpha(x, y_\alpha)$ such that $\phi_\alpha(x, a_{\alpha,0}) \in p$ and $\{\phi_\alpha(x, a_{\alpha,i})\}_{i<\omega}$ is inconsistent.

(1) is possible by saturation of M. But then by Lemma 38, $(\bar{a}_\alpha)_{\alpha<|T|^+}$ are almost mutually indiscernible. $\square$

3.2. **Dependent dividing.**

Definition 40. We say that T has *dependent dividing* if given $M \preceq N$ and $p(x) \in S(N)$ dividing over M, then there is a dependent formula $\phi(x, y)$ and $c \in N$ such that $\phi(x, c) \in p$ and $\phi(x, c)$ divides over M.

Proposition 41. (1) *If T has dependent dividing, then it is* NTP_2*.*

(2) *If T has simple dividing, then it is simple.*

Proof. (1) In fact we will only use that dividing is always witnessed by an instance of an NTP_2 formula. Assume that T has TP_2 and let $\phi(x, y)$ witness this. Let T_{Sk} be a Skolemization of T, $\phi(x, y)$ still has TP_2 in T_{Sk}. Then as in the proof of Theorem 36, for any κ we can find $(b_i)_{i<\kappa}$, a and M such that $a \models \{\phi(x, b_i)\}_{i<\kappa}$, $\phi(x, b_i)$ divides over M and $\mathrm{tp}\,(b_i/b_{<i}M)$ has a global heir-coheir over M, all in the sense of T_{Sk}. Taking $M_i = \mathrm{Sk}(Mb_i) \models T$, and now working in T, we still have that $a \not\downarrow^{d}_{M} M_i$ and $M_i \downarrow^{\mathrm{ist}}_{M} M_{<i}$ (as $\mathrm{tp}(M_i/M_{<i}M)$ still has a global heir-coheir over M). But then for each i we find some $d_i \in M_i$ and NTP_2 formulas $\phi_i(x, y_i) \in L$ such that $a \models \{\phi_i(x, d_i)\}$ and $\phi_i(x, d_i)$ divides over M, witnessed by $\bar{d}_i$ starting with d_i. We may assume that $\phi_i = \phi'$, and this contradicts ϕ' being NTP_2.

(2) Similar argument shows that if T has simple dividing, then it is simple.

□

Of course, if T is NIP, then it has dependent dividing, and for simple theories it is equivalent to the stable forking conjecture. It is natural to ask if every NTP_2 theory T has dependent dividing.

3.3. **Forking and dividing inside an NTP_2 type.**

Definition 42. A partial type $p(x)$ over C is said to be NTP_2 if the following does not exist: $(\bar{a}_\alpha)_{\alpha<\omega}$, $\phi(x, y)$ and $k < \omega$ such that $\{\phi(x, a_{\alpha i})\}_{i<\omega}$ is k-inconsistent for every $\alpha < \omega$ and $\left\{\phi(x, a_{\alpha f(\alpha)})\right\}_{\alpha<\omega} \cup p(x)$ is consistent for every $f : \omega \to \omega$. Of course, T is NTP_2 if and only if every partial type is NTP_2. Also notice that if $p(x)$ is NTP_2, then every extension of it is NTP_2 and that $q((x_i)_{i<\kappa}) = \bigcup_{i<\kappa} p(x_i)$ is NTP_2 (follows from Theorem 11).

For the later use we will need a generalization of the results from [CK12] working inside a partial NTP_2 type, and with no assumption on the theory.

Lemma 43. *Let $p(x)$ be an* NTP_2 *type over M. Assume that $p(x) \cup \{\phi(x, a)\}$ divides over M, then there is a global coheir $q(x)$ extending* $\mathrm{tp}(a/M)$ *such that $p(x) \cup \{\phi(x, a_i)\}_{i<\omega}$ is inconsistent for any sequence $(a_i)_{i<\omega}$ with $a_i \models q|_{a_{<i}M}$.*

Proof. The proof of [CK12, Lemma 3.12] goes through. □

Lemma 44. *Assume that* $\operatorname{tp}(a_i/C) = p(x)$ *for all* i *and that* $\operatorname{tp}(a_i/a_{<i}C)$ *has a strictly invariant extension to* $p(\mathbb{M}) \cup C$. *Then there are mutually* C*-indiscernible* $\left(\bar{b}_i\right)_{i<\kappa}$ *such that* $\bar{b}_i \equiv_{a_i C} \bar{a}_i$.

Proof. The assumption is sufficient for the proof of Lemma 31 to work. $\square$

Lemma 45. *Let* $p(x)$ *over* M *be* NTP_2, $a \in p(\mathbb{M})$, $c \in M$ *and assume that* $p(x) \cup \{\phi(x, ac)\}$ *divides over* M. *Assume that* $\operatorname{tp}(a/M)$ *has a strictly invariant extension* $p'(y) \in S(p(\mathbb{M}))$. *Then for any* $(a_i)_{i<\omega}$ *such that* $a_i \models p'|_{a_{<i}M}$, $p(x) \cup \{\phi(x, a_i c)\}_{i<\omega}$ *is inconsistent.*

Proof. Let $(\bar{a}_0 c)$ with $a_{0,0} = a_0$ be an M-indiscernible sequence witnessing that $p(x) \cup \{\phi(x, a_0 c)\}$ divides over M. Let $\bar{a}_i$ be its image under an M-automorphism sending a_0 to a_i. By Lemma 31(2) we can find $\left(\bar{b}_i\right)_{i<\omega}$ mutually indiscernible over M and with $\bar{b}_i \equiv_{a_i M} \bar{a}_i$. By the choice of $\bar{b}_i$'s and compactness, there is some $\psi(x) \in p(x)$ such that $\{\psi(x) \wedge \phi(x, b_{i,j} c)\}_{j<\omega}$ is k-inconsistent for all $i < \omega$. It follows that $p(x) \cup \{\phi(x, a_i c)\}_{i<\omega}$ is inconsistent as p is NTP_2. $\square$

We need a version of the Broom lemma localized to an NTP_2 type.

Lemma 46. *Let* $p(x)$ *be an* NTP_2 *type over* M *and* $p'(x)$ *be a partial global type invariant over* M. *Suppose that* $p(x) \cup p'(x) \vdash \bigvee_{i<n} \phi_i(x, c)$ *and each* $\phi_i(x, c)$ *divides over* M. *Then* $p(x) \cup p'(x)$ *is inconsistent.*

Proof. Follows from the proof of [CK12, Lemma 3.1]. $\square$

Corollary 47. *Let* $p(x)$ *be an* NTP_2 *type over* M *and* $a \in p(\mathbb{M})$. *Then* $\operatorname{tp}(a/M)$ *has a strictly invariant extension* $p'(x) \in S(p(\mathbb{M}) \cup M)$.

Proof. Following the proof of [CK12, Proposition 3.7] but using Lemma 46 in place of the Broom lemma. $\square$

And finally,

Proposition 48. *Let* $p(x)$ *be an* NTP_2 *type,* $a \in p(\mathbb{M}) \cup M$ *and assume that* $\{\phi(x, a)\} \cup p(x)$ *does not divide. Then there is* $p'(x) \in S(p(\mathbb{M}) \cup M)$ *which does not divide over* M *and* $\{\phi(x, a)\} \cup p(x) \subset p'(x)$.

Proof. By compactness, it is enough to show that if $p(x) \cup \{\phi(x, ac)\} \vdash \bigvee_{i<n} \phi_i(x, a_i c_i)$ with $a, a_i \in p(\mathbb{M})$ and $c, c_i \in M$, then $p(x) \cup \{\phi_i(x, a_i c_i)\}$ does not divide for some $i < n$. As in the proof of [CK12, Corollary 3.16], let $\left(a^j a_0^j \ldots a_{n-1}^j\right)_{j<\omega}$ be a strict Morley sequence in $\operatorname{tp}(a a_0 \ldots a_{n-1})$, which exists by Lemma 47. Notice that $\left(a^j c a_0^j c_0 \ldots a_{n-1}^j c_{n-1}\right)_{j<\omega}$ is still indiscernible over M. Then $p(x) \cup \left\{\phi(x, a^j c)\right\}_{j<\omega}$ is consistent, which implies that $p(x) \cup \left\{\phi_i(x, a_i^j c_i)\right\}_{j<\omega}$ is consistent for some $i < n$. But then by Lemma 45, $p(x) \cup \{\phi_i(x, a_i c_i)\}$ does not divide over M — as wanted. $\square$

4. NIP TYPES

Let T be an arbitrary theory.

Definition 49. (1) A partial type $p(x)$ over C is called NIP if there is no $\phi(x,y) \in L$, $(a_i)_{i\in\omega}$ with $a_i \models p(x)$ and $(b_s)_{s\subseteq\omega}$ such that $\models \phi(a_i,b_s) \Leftrightarrow i \in s$.

(2) The roles of a's and b's in the definition are interchangeable. It is easy to see that any extension of an NIP type is again NIP, and that the type of several realizations of an NIP type is again NIP.

(3) $p(x)$ is NIP $\Leftrightarrow$ $\mathrm{dprk}(p) < |T|^+ \Leftrightarrow \mathrm{dprk}(p) < \infty$ (see Definition 20).

Lemma 50. *Let $p(x)$ be an* NIP *type.*

(1) *Let $\bar{a} = (a_\alpha)_{\alpha<\kappa}$ be an indiscernible sequence over A with a_α from $p(\mathbb{M})$, and c be arbitrary. If $\kappa = (|a_\alpha| + |c|)^+$, then some non-empty end segment of $\bar{a}$ is indiscernible over Ac.*

(2) *Let $(\bar{a}_\alpha)_{\alpha<\kappa}$ be mutually indiscernible (over $\emptyset$), with $\bar{a}_\alpha = (a_{\alpha i})_{i<\lambda}$ from $p(\mathbb{M})$. Assume that $\bar{a} = (a_{0i}a_{1i}...)_{i<\lambda}$ is indiscernible over A. Then $(\bar{a}_\alpha)_{\alpha<\kappa}$ is mutually indiscernible over A.*

Standard proofs of the corresponding results for NIP theories go through, see e.g. [Adl08].

4.1. Dp-rank of a type is always witnessed by an array of its realizations.

In [KS12] Kaplan and Simon demonstrate that inside an NTP_2 theory, dp-rank of a type can always be witnessed by mutually indiscernible sequences of realizations of the type. In this section we show that the assumption that the theory is NTP_2 can be omitted, thus proving the following general theorem with no assumption on the theory.

Theorem 51. *Let $p(x)$ be an* NIP *partial type over C, and assume that* $\mathrm{dprk}(p) \geq \kappa$. *Then there is $C' \supseteq C$, $b \models p(x)$ and $(\bar{a}_\alpha)_{\alpha<\kappa}$ with $\bar{a}_\alpha = (a_{\alpha i})_{i<\omega}$ such that:*

- *$a_{\alpha i} \models p(x)$ for all α, i*
- *$(\bar{a}_\alpha)_{\alpha<\kappa}$ are mutually indiscernible over C'*
- *None of $\bar{a}_\alpha$ is indiscernible over bC'.*
- *$|C'| \leq |C| + \kappa$.*

Corollary 52. *It follows that dp-rank of a 1-type is always witnessed by mutually indiscernible sequences of singletons.*

We will use the following result from [CS10, Proposition 1.1]:

Fact 53. *Let $p(x)$ be a (partial)* NIP *type, $A \subseteq p(\mathbb{M})$ and $\phi(x,c)$ given. Then there is $\theta(x,d)$ with $d \in p(\mathbb{M})$ such that:*

(1) $\theta(A,d) = \phi(A,c)$,

(2) $\theta(x,d) \cup p(x) \to \phi(x,c)$.

We begin by showing that the burden of a dependent type can always be witnessed by mutually indiscernible sequences from the set of its realizations.

Lemma 54. *Let $p(x)$ be a dependent partial type over C of burden $\geq \kappa$. Then we can find $\left(\bar{d}_\alpha\right)_{\alpha<\kappa}$ witnessing it, mutually indiscernible over C and with $\bar{d}_i \subseteq p(\mathbb{M}) \cup C$.*

Proof. Let λ be large enough compared to $|C|$. Assume that $\mathrm{bdn}(p) \geq \kappa$, then by compactness we can find $\left(\bar{b}_\alpha, \phi_\alpha(x,y_\alpha), k_\alpha\right)_{i<n}$ such that $\bar{b}_\alpha = (b_{\alpha i})_{i<\lambda}$, $\{\phi_\alpha(x,b_{\alpha i})\}_{\alpha<\kappa}$ is k_α-inconsistent and $p(x) \cup \left\{\phi_\alpha(x, b_{\alpha f(\alpha)})\right\}_{i<n}$ is consistent for every $f : \kappa \to \lambda$, let a_f realize it. Set $A = \{a_f\}_{f\in\lambda^\kappa} \subseteq p(\mathbb{M})$.

By Fact 53, let $\theta_{\alpha i}(x, d_{\alpha i})$ be an honest definition of $\phi_\alpha(x, b_{\alpha i})$ over A (with respect to $p(x)$), with $d_{\alpha i} \in p(\mathbb{M})$. As λ is very large, we may assume that $\theta_{\alpha i} = \theta_\alpha$.

Now, as $\theta_\alpha(x, d_{\alpha i}) \cup p(x) \to \phi_\alpha(x, b_{\alpha i})$, it follows that there is some $\psi_\alpha(x,c) \in p$ such that letting $\chi_\alpha(x, y_1 y_2) = \theta_\alpha(x,y_1) \wedge \psi_\alpha(x,y_2)$, $\{\chi(x, d_{\alpha i} c_\alpha)\}_{i<\omega}$ is k_α-inconsistent.

On the other hand, $\left\{\chi_\alpha(x, d_{\alpha f(\alpha)} c_\alpha)\right\}_{\alpha<\kappa} \cup p(x)$ is consistent, as the corresponding a_f realizes it. Thus this array still witnesses that burden of p is at least κ. □

We will also need the following lemma.

Lemma 55. *Let $p(x)$ be an* NIP *type over $M \models T$*

(1) *Assume that $a \in p(\mathbb{M}) \cup M$ and $\phi(x,a)$ does not divide over M, then there is a type $q(x) \in S(p(\mathbb{M}) \cup M)$ invariant under M-automorphisms and with $\phi(x,a) \in q$.*
(2) *Let $p'(x) \supset p(x)$ be an M invariant type such that $p^{(\omega)}$ is an heir-coheir over M. If $(a_i)_{i<\omega}$ is a Morley sequence in p' and indiscernible over bM with $b \in p(\mathbb{M})$, then* $\mathrm{tp}(b/MI)$ *has an M-invariant extension in $S(p(\mathbb{M}) \cup M)$.*

Proof. (1) As NIP type is in particular an NTP$_2$ type, by Lemma 48 we find a type $q(x) \in S(p(\mathbb{M}))$ which doesn't divide over M and such that $\phi(x,a) \in q$. It is enough to show that $q(x)$ is Lascar-invariant over M. Assume that we have an M-indiscernible sequence $(a_i)_{i<\omega}$in $p(\mathbb{M})$ such that $\phi(x,a_0) \wedge \neg\phi(x,a_1) \in q$. But then $\{\phi(x,a_{2i}) \wedge \phi(x,a_{2i+1})\}_{i<\omega}$ is inconsistent, so q divides over M — a contradiction. Easy induction shows the same for a_0 and a_1 at Lascar distance n.

(2) By Lemma 45 and (1). □

Now for the *proof of Theorem 51.* The point is that first the array witnessing dp-rank of our type $p(x)$ can be dragged inside the set of realizations of p by Lemma 54. Then, combined with the use of Proposition 55 instead of the unrelativized version, the proof of Kaplan and Simon [KS12, Section 3.2] goes through working inside $p(\mathbb{M})$.

Problem 56. Is the analogue of Lemma 54 true for the burden of an arbitrary type in an NTP$_2$ theory?

We include some partial observations to justify it.

Proposition 57. *The answer to the Problem 54 is positive in the following cases:*

1. *T satisfies dependent forking (so in particular if T is* NIP*).*
2. *T is simple.*

Proof. (1): Recall that if $\mathrm{bdn}(p) \geq \kappa$, then we can find $(b_i)_{i<\kappa}$, $a \models p$ and $M \supseteq C$ such that $a \not\mathrel{\unicode{x2AEB}}^{d}_{M} b_i$ and $b_i \mathrel{\unicode{x2AEB}}^{\mathrm{ist}}_{M} b_{<i}$. Notice that $p(x)$ still has the same burden in the sense of a Skolemization T^{Sk}. Choose inductively $M_i \supseteq M \cup b_i$ such that $M_i \mathrel{\unicode{x2AEB}}^{\mathrm{ist}}_{M} b_{<i}$, let $M_i = Sk(M \cup b_i)$. Let $\phi(x, b_i)$ be witness this dividing with $\phi(x, y)$ an NIP formula, we can make $\bar{b}_i$ mutually indiscernible. Now the proof of Lemma 54 goes through.

(2): Let $p(x) \in S(A)$, $a \models p(x)$ and let $(b_i)_{i<\kappa}$ independent over A, with $a \not\mathrel{\unicode{x2AEB}}_A b_i$. Without loss of generality $A = \emptyset$. Consider $\mathrm{tp}(a/b_0)$ and take $I = (a_i)_{i<|T|^+}$ such that $a \frown I$ is a Morley sequence in it. By extension and automorphism we may assume $b_{>0} \mathrel{\unicode{x2AEB}}_{ab_0} I$, together with $a \mathrel{\unicode{x2AEB}}_{b_0} I$ implies $b_{>0} \mathrel{\unicode{x2AEB}}_{b_0} I$, thus $b_{>0} \mathrel{\unicode{x2AEB}} I$ (as $b_{>0} \mathrel{\unicode{x2AEB}} b_0$).

Assume that I is a Morley sequence over $\emptyset$, then by simplicity $a_i \mathrel{\unicode{x2AEB}} b_0$ for some i, contradicting $a_i \equiv_{b_0} a$ and $a \not\mathrel{\unicode{x2AEB}} b_0$. Thus by indiscernibility $a \not\mathrel{\unicode{x2AEB}} a_{<n}$ for some n, while $\{a_{<n}\} \cup b_{>0}$ is an independent set.

Repeating this argument inductively and using the fact that the burden of a type in a simple theory is the supremum of the weights of its completions (Fact 24) allows to conclude. $\square$

4.2. **NIP types inside an** NTP$_2$ **theory.** We give a characterization of NIP types in NTP$_2$ theories in terms of the number of non-forking extensions of its completions.

Theorem 58. *Let T be* NTP$_2$*, and let $p(x)$ be a partial type over C. The following are equivalent:*

1. *p is* NIP*.*
2. *Every $p' \supseteq p$ has boundedly many global non-forking extensions.*

Proof. (1)$\Rightarrow$(2): A usual argument shows that a non-forking extension of an NIP type is in fact Lascar-invariant (see Lemma 55), thus there are only boundedly many such.

(2)$\Rightarrow$(1): Assume that $p(x)$ is not NIP, that is there are $I = (b_i)_{i\in\omega}$ such that such that for any $s \subseteq \omega$, $p_s(x) = p(x) \cup \{\phi(x, b_i)\}_{i\in s} \cup \{\neg\phi(x, b_i)\}_{i\notin s}$ is consistent. Let $q(y)$ be a global non-algebraic type finitely satisfiable in I. Let $M \supseteq IC$ be some $|IC|^+$-saturated model. It follows that $q^{(\omega)}$ is a global heir-coheir over M by Lemma 30. Take an arbitrary cardinal κ, and let $J = (c_i)_{i\in\kappa}$ be a Morley sequence in q over M. We claim that for any $s \subseteq \kappa$, $p_s(x)$ does not divide over M. First notice that $p_s(x)$ is consistent for any s, as $\mathrm{tp}(J/M)$ is finitely satisfiable in I. But as for

any $k<\omega$, $(c_{ki}c_{ki+1}...c_{k(i+1)-1})_{i<\omega}$ is a Morley sequence in $q^{(k)}$, together with Fact33 this implies that $p_s(x)|_{c_0...c_{k-1}}$ does not divide over M for any $k<\omega$, thus by indiscernibility of J, $p_s(x)$ does not divide over M, thus has a global non-forking extension by Fact 33.

As there are only boundedly many types over M, there is some $p' \in S(M)$ extending p, with unboundedly many global non-forking extensions. □

Remark 59. (2)⇒(1) is just a localized variant of an argument from [CKS12].

5. Simple types

5.1. **Simple and co-simple types.**

Simple types, to the best of our knowledge, were first defined in [HKP00, §4] in the form of (2).

Definition 60. We say that a partial type $p(x) \in S(A)$ is *simple* if it satisfies any of the following equivalent conditions:

(1) There is no $\phi(x,y)$, $(a_\eta)_{\eta\in\omega^{<\omega}}$ and $k<\omega$ such that: $\{\phi(x,a_{\eta i})\}_{i<\omega}$ is k-inconsistent for every $\eta\in\omega^{<\omega}$ and $\{\phi(x,a_{\eta\restriction i})\}_{i<\omega}\cup p(x)$ is consistent for every $\eta\in\omega^\omega$.

(2) Local character: If $B\supseteq A$ and $p(x)\subseteq q(x)\in S(B)$, then $q(x)$ does not divide over AB' for some $B'\subseteq B$, $|B'|\le|T|$.

(3) Kim's lemma: If $\{\phi(x,b)\}\cup p(x)$ divides over $B\supseteq A$ and $(b_i)_{i<\omega}$ is a Morley sequence in $\mathrm{tp}(b/B)$, then $p(x)\cup\{\phi(x,b_i)\}_{i<\omega}$ is inconsistent.

(4) Bounded weight: Let $B\supseteq A$ and $\kappa\ge\beth_{(2^{|B|})^+}$. If $a\models p(x)$ and $(b_i)_{i<\kappa}$ is such that $b_i \mathop{\smile\!\!\!\!\!|}^f_B b_{<i}$, then $a \mathop{\smile\!\!\!\!\!|}^d_B b_i$ for some $i<\kappa$.

(5) For any $B\supseteq A$, if $b \mathop{\smile\!\!\!\!\!|}^f_B a$ and $a\models p(x)$, then $a \mathop{\smile\!\!\!\!\!|}^d_B b$.

Proof.

(1)⇒(2): Assume (2) fails, then we choose $\phi_\alpha(x,b_\alpha)\in q(x)$ k_α-dividing over $A\cup B_\alpha$, with $B_\alpha=\{b_\beta\}_{\beta<\alpha}\subseteq B$, $|B_\alpha|\le|\alpha|$ by induction on $\alpha<|T|^+$. Then w.l.o.g. $\phi_\alpha=\phi$ and $k_\alpha=k$. Now construct a tree in the usual manner, such that $\{\phi(x,a_{\eta i})\}_{i<\omega}$ is inconsistent for any $\eta\in\omega^{<\omega}$ and $\{\phi(x,a_{\eta|i})\}_{i<\omega}\cup p(x)$ is consistent for any $\eta\in\omega^\omega$.

(2)⇒(3): Let $I=(|T|^+)^*$, and $(b_i)_{i\in I}$ be Morley over B in $\mathrm{tp}(b/B)$. Assume that $a\models p(x)\cup\{\phi(x,b_i)\}_{i\in I}$. By (2), $\mathrm{tp}(a/(b_i)_{i\in I}B)$ does not divide over $B(b_i)_{i\in I_0}$ for some $I_0\subseteq I$, $|I_0|\le|T|$. Let $i_0\in I$, $i_0<I_0$. Then $(b_i)_{i\in I_0} \mathop{\smile\!\!\!\!\!|}^f_B b_{i_0}$, and thus $\phi(x,b_{i_0})$ divides over BI_0 - a contradiction.

(3)⇒(4): Assume not, then by Erdös-Rado and finite character find a Morley sequence over B and a formula $\phi(x,y)$ such that $\models\phi(a,b_i)$ and $\phi(x,b_i)$ divides over B, contradiction to (3).

(4)⇒(5): For κ as in (4), let $I=(b_i)_{i<\kappa}$ be a Morley sequence over B, indiscernible over Ba and with $b_0=b$. By (4), $a \mathop{\smile\!\!\!\!\!|}^d_B b_i$ for some $i<\kappa$, and so $a \mathop{\smile\!\!\!\!\!|}^d_B b$ by indiscernibility.

(5)⇒(1): Let $(b_\eta)_{\eta\in\omega^{<\omega}}$ witness the tree property of $\phi(x,y)$, such that $\{\phi(x,b_{\eta|i})\}_{i<\omega}\cup p(x)$ is consistent for every $\eta\in\omega^\omega$. Then by Ramsey and compactness we can find $(b_i)_{i\leq\omega}$ indiscernible over a, $\models\phi(a,b_i)$ and $\phi(x,b_i)$ divides over $b_{<i}A$. Taking $B=A\cup\{b_i\}_{i<\omega}$ we see that $a\not\downarrow^d_B b_\omega$, while $b_\omega\downarrow^f_B a$ (as it is finitely satisfiable in B by indiscernibility) - a contradiction to (5).

□

Remark 61. Let $p(x)\in S(A)$ be simple.

(1) Any $q(x)\supseteq p(x)$ is simple.
(2) Let $p(x)\in S(A)$ be simple and $C\subseteq p(\mathbb{M})$. Then $\mathrm{tp}(C/A)$ is simple.

Proof. (1): Clear, for example by (1) from the definition.

(2): Let $C=(c_i)_{i\leq n}$, and we show that for any $B\supseteq A$, if $b\downarrow^f_B C$, then $C\downarrow^d_B b$ by induction on the size of C. Notice that $b\downarrow^f_{Bc_{<n}} c_n$ and $c_n\models p$, thus $c_n\downarrow^d_{Bc_{<n}} b$. By the inductive assumption $c_{<n}\downarrow^d_B b$, thus $c_{\leq n}\downarrow^d_B b$. □

We give a characterization in terms of local ranks.

Proposition 62. *The following are equivalent:*

(1) *$p(x)$ is simple in the sense of Definition 60.*
(2) *$D(p,\Delta,k)<\omega$ for any finite Δ and $k<\omega$.*

Proof. Standard proof goes through. □

Lemma 63. *Let $p(x)\in S(A)$ be simple, $a\models p(x)$ and $B\supseteq A$ arbitrary. Then $a\downarrow^f_{B_0} B$ for some $|B_0|\leq|T|^+$.*

Proof. Standard proof using ranks goes through. □

It follows that in the Definition 60 we can replace everywhere "dividing" by "forking".

Lemma 64. *Let $p(x)\in S(A)$ be simple. If A is an extension base, then $\{\phi(x,c)\}\cup p(x)$ forks over A if and only if it divides over A.*

Proof. Assume that $\{\phi(x,c)\}\cup p(x)$ does not divide over A, but $\{\phi(x,c)\}\cup p(x)\vdash\bigvee_{i<n}\phi_i(x,c_i)$ and each of $\phi_i(x,c_i)$ divides over A. As A is an extension base, let $(c_ic_{0,i}...c_{n-1,i})$ be a Morley sequence in $\mathrm{tp}(cc_0...c_{n-1}/A)$. As $p(x)\cup\{\phi(x,c)\}$ does not divide over A, let $a\models p(x)\cup\{\phi(x,c_i)\}$, but then $p(x)\cup\{\phi_i(x,c_{i,j})\}_{j<\omega}$ is consistent for some $i<n$, contradicting Kim's lemma. □

Problem 65. Let $q(x)$ be a non-forking extension of a complete type $p(x)$, and assume that $q(x)$ is simple. Does it imply that $p(x)$ is simple?

Unlike stability or NIP, it is possible that $\phi(x,y)$ does not have the tree property, while $\phi^*(x',y')=\phi(y',x')$ does. This forces us to define a dual concept.

Definition 66. A partial type $p(x)$ over A is *co-simple* if it satisfies any of the following equivalent properties:

(1) No formula $\phi(x,y)\in L(A)$ has the tree property witnessed by some $(a_\eta)_{\eta\in\omega^{<\omega}}$ with $a_\eta\subseteq p(\mathbb{M})$.

(2) Every type $q(x)\in S(BA)$ with $B\subseteq p(\mathbb{M})$ does not divide over AB' for some $B'\subseteq B$, $|B'|\leq(|A|+|T|)^+$.

(3) Let $(a_i)_{i<\omega}\subseteq p(\mathbb{M})$ be a Morley sequence over BA, $B\subseteq p(\mathbb{M})$ and $\phi(x,y)\in L(A)$. If $\phi(x,a_0)$ divides over BA then $\{\phi(x,a_i)\}_{i<\omega}$ is inconsistent.

(4) Let $B\subseteq p(\mathbb{M})$ and $\kappa\geq\beth_{\left(2^{|B|+|A|}\right)^+}$. If $(b_i)_{i<\kappa}\subseteq p(\mathbb{M})$ is such that $b_i\mathop{\smile\!\!\!\!\!\mid}^{f}_{AB}b_{<i}$ and a arbitrary, then $a\mathop{\smile\!\!\!\!\!\mid}^{d}_{AB}b_i$ for some $i<\kappa$.

(5) For $B\subseteq p(\mathbb{M})$, if $a\models p$ and $a\mathop{\smile\!\!\!\!\!\mid}^{f}_{AB}b$, then $b\mathop{\smile\!\!\!\!\!\mid}^{d}_{AB}a$.

Proof. Similar to the proof in Definition 60. $\square$

Remark 67. It follows that if $p(x)$ is a co-simple type over A and $B\subseteq p(\mathbb{M})$, then any $q(x)\in S(AB)$ extending p is co-simple (while adding the parameters from outside of the set of solutions of p may ruin co-simplicity).

It is easy to see that T is simple $\Leftrightarrow$ every type is simple $\Leftrightarrow$ every type is co-simple. What is the relation between simple and co-simple in general?

Example 68. There is a co-simple type over a model which is not simple.

Proof. Let T be the theory of an infinite triangle-free random graph, this theory eliminates quantifiers. Let $M\models T$, $m\in M$ and consider $p(x)=\{xRm\}\cup\{\neg xRa\}_{a\in M\setminus\{m\}}$ - a non-algebraic type over M. As there can be no triangles, if $a,b\models p(x)$ then $\neg aRb$. It follows that for any $A\subseteq p(\mathbb{M})$ and any B, $B\not\mathop{\smile\!\!\!\!\!\mid}^{d}_{M}A\Leftrightarrow B\cap A\neq\emptyset$. So $p(x)$ is co-simple, for example by checking the bounded weight (Definition 66(4)).

For each $\alpha<\omega$, take $\left(b'_{\alpha,i}b''_{\alpha,i}\right)_{i<\omega}$ such that $b'_{\alpha,i}Rb''_{\alpha,j}$ for all $i\neq j$, and no other edges between them or to elements of M. Then $\left\{xRb'_{\alpha,i}\wedge xRb''_{\alpha,i}\right\}_{i<\omega}$ is 2-inconsistent for every α, while $p(x)\cup\left\{xRb'_{\alpha,\eta(\alpha)}\wedge xRb''_{\alpha,\eta(\alpha)}\right\}_{\alpha<\omega}$ is consistent for every $\eta:\omega\to\omega$. Thus $p(x)$ is not simple by Definition 60(1). $\square$

However, this T has TP_2.

Problem 69. Is there a simple, non co-simple type in an arbitrary theory?

5.2. **Simple types are co-simple in NTP$_2$ theories.** In this section we assume that T is NTP$_2$ (although some lemmas remain true without this restriction). In particular, we will write $\downfree$ to denote non-forking/non-dividing when working over an extension base as they are the same by Fact 33(3).

Lemma 70. *Weak chain condition: Let A be an extension base, $p(x) \in S(A)$ simple. Assume that $a \models p(x)$, $I = (b_i)_{i<\omega}$ is a Morley sequence over A and $a \downfree_A b_0$. Then there is an aA-indiscernible $J \equiv_{Ab_0} I$ satisfying $a \downfree_A J$.*

Proof. Let $a \models \phi(x, b_0)$, then $\{\phi(x, b_0)\} \cup p(x)$ does not divide over A.

Claim. $\{\phi(x, b_0) \wedge \phi(x, b_1)\} \cup p(x)$ does not divide over A.

Proof. As $p(x)$ satisfies Definition 60(3), $(b_{2i}b_{2i+1})_{i<\omega}$ is a Morley sequence over A and $\{\phi(x, b_i)\}_{i<\omega} \cup p(x)$ is consistent. □

By iterating the claim and compactness, we conclude that $\bigcup_{i<\omega} p(x, b_i)$ does not divide over A, where $p(x, b_0) = \mathrm{tp}(a/b_0)$. As A is an extension base and forking equals dividing, there is $a' \models \bigcup_{i<\omega} p(x, b_i)$ satisfying $a' \downfree_A I$. By Ramsey, compactness and the fact that $a'b_i \equiv_A ab_0$ we find a sequence as wanted. □

Remark 71. If fact, in [BYC] we demonstrate that in an NTP$_2$ theory this lemma holds over extension bases with I just an indiscernible sequence, not necessarily Morley.

Lemma 72. *Let A be an extension base, $p \in S(A)$ simple. For $i < \omega$, Let $\bar{a}_i$ be a Morley sequence in $p(x)$ over A starting with a_i, and assume that $(a_i)_{i<\omega}$ is a Morley sequence in $p(x)$. Then we can find $\bar{b}_i \equiv_{Aa_i} \bar{a}_i$ such that $(\bar{b}_i)_{i<\omega}$ are mutually indiscernible over A.*

Proof. W.l.o.g. $A = \emptyset$.

First observe that by simplicity of p, $\{a_i\}_{i<\omega}$ is an independent set.

For $i < \omega$, we choose inductively $\bar{b}_i$ such that:

(1) $\bar{b}_i \equiv_{a_i} \bar{a}_i$
(2) $\bar{b}_i$ is indiscernible over $a_{>i}\bar{b}_{<i}$
(3) $a_{>i+1}\bar{b}_{\leq i} \downfree a_{i+1}$
(4) $a_{\geq i+1} \downfree \bar{b}_{\leq i}$

Base step: As $a_{>0} \downfree a_0$ and $\mathrm{tp}(a_{>0})$ is simple by Remark 61 and Lemma 70, we find an $a_{>0}$-indiscernible $\bar{b}_0 \equiv_{a_0} \bar{a}_0$ with $a_{>0} \downfree \bar{b}_0$.

Induction step: Assume that we have constructed $\bar{b}_0, ..., \bar{b}_{i-1}$. By (3) for $i-1$ it follows that $a_{>i}\bar{b}_{<i} \downfree a_i$. Again by Remark 61 and Lemma 70 we find an $a_{>i}\bar{b}_{<i}$-indiscernible sequence $\bar{b}_i \equiv_{a_i} \bar{a}_i$ such that $a_{>i}\bar{b}_{<i} \downfree \bar{b}_i$.

We check that it satisfies (3): As all tuples are inside $p(\mathbb{M})$, we can use symmetry, transitivity and $\mathop{\smile\!\!\!\!\!\mid}^{d} = \mathop{\smile\!\!\!\!\!\mid}^{f}$ freely. And so, $a_{>i+1}a_{i+1}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\mid} \bar{b}_i \Rightarrow a_{>i+1}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\mid}_{a_{i+1}} \bar{b}_i + a_{>i+1}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\mid} a_{i+1}$(as $a_{>i+1} \mathop{\smile\!\!\!\!\!\mid} a_{i+1}$ and $\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\mid} a_{\geq i+1}$ by (4) for $i-1$) $\Rightarrow a_{>i+1}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\mid} \bar{b}_i a_{i+1} \Rightarrow a_{>i+1}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\mid}_{\bar{b}_i} a_{i+1} + \bar{b}_i \mathop{\smile\!\!\!\!\!\mid} a_{i+1} \Rightarrow a_{>i+1}\bar{b}_{\leq i} \mathop{\smile\!\!\!\!\!\mid} a_{i+1}$.

We check that it satisfies (4): As $a_{>i}\bar{b}_{<i} \mathop{\smile\!\!\!\!\!\mid} \bar{b}_i \Rightarrow a_{>i} \mathop{\smile\!\!\!\!\!\mid}_{\bar{b}_{<i}} \bar{b}_i + a_{>i} \mathop{\smile\!\!\!\!\!\mid} \bar{b}_{<i}$ by (4) for $i-1 \Rightarrow a_{>i} \mathop{\smile\!\!\!\!\!\mid} \bar{b}_{\leq i}$.

Having chosen $(\bar{b}_i)_{i<\omega}$ we see that they are almost mutually indiscernible by (1) and (2). Conclude by Lemma 3. $\square$

Lemma 73. *Let T be NTP$_2$, A an extension base and $p(x) \in S(A)$ simple. Assume that $\phi(x,a)$ divides over A, with $a \models p(x)$. Then there is a Morley sequence over A witnessing it.*

Proof. As A is an extension base, let $M \supseteq A$ be such that $M \mathop{\smile\!\!\!\!\!\mid}_A^f a$. Then $\phi(x,a)$ divides over M. By Fact 33(1), there is a Morley sequence $(a_i)_{i<\omega}$ over M witnessing it (in particular $(a_i)_{i<\omega} \subseteq p(\mathbb{M})$). We show that it is actually a Morley sequence over A. Indiscernibility is clear, and we check that $a_i \mathop{\smile\!\!\!\!\!\mid}_A a_{<i}$ by induction. As $a_i \mathop{\smile\!\!\!\!\!\mid}_M a_{<i}$, $a_{<i} \mathop{\smile\!\!\!\!\!\mid}_M a_i$ by simplicity of $\mathrm{tp}(a_{<i}/M)$. Noticing that $M \mathop{\smile\!\!\!\!\!\mid}_A a_i$, we conclude $a_{<i} \mathop{\smile\!\!\!\!\!\mid}_A a_i$, so again by simplicity $a_i \mathop{\smile\!\!\!\!\!\mid}_A a_{<i}$. $\square$

Proposition 74. *Let T be NTP$_2$, A an extension base and $p(x) \in S(A)$ simple. Assume that $a \models p$ and $a \mathop{\smile\!\!\!\!\!\mid}_A^f b$. Then $b \mathop{\smile\!\!\!\!\!\mid}_A^d a$.*

Proof. Assume that there is $\phi(x,a) \in L(Aa)$ such that $\models \phi(b,a)$ and $\phi(x,a)$ divides over A. Let $(a_i)_{i<\omega}$ be a Morley sequence over A starting with a. Assume that $\{\phi(x,a_i)\}_{i<\omega}$ is consistent. Let $\bar{a}_0$ be a Morley sequence witnessing that $\phi(x,a_0)$ k-divides over A (exists by Lemma 73), and let $\bar{a}_i$ be its image under an A-automorphism sending a_0 to a_i. By Lemma 72, we find $\bar{a}_i' \equiv_{a_i A} \bar{a}_i$, such that $(\bar{a}_i')_{i<\omega}$ are mutually indiscernible. But then we have that $\{\phi(x,a_{i,\eta(i)})\}_{i<\omega}$ is consistent for any $\eta \in \omega^\omega$, while $\{\phi(x,a_{i,j})\}_{j<\omega}$ is k-inconsistent for any $i<\omega$ — contradiction to NTP$_2$.

Now let $(a_i)_{i<\omega}$ be a Morley sequence over A starting with a and indiscernible over Ab. Then clearly $b \models \{\phi(x,a_i)\}_{i<\omega}$ for any $\phi(x,a) \in \mathrm{tp}(b/aA)$, so by the previous paragraph $b \mathop{\smile\!\!\!\!\!\mid}_A^d a$. $\square$

Lemma 75. *Let $p(x)$ be a partial type over A. Assume that $p(x)$ is not co-simple over A. Then there is some $M \supseteq A$, $a \models p(x)$ and b such that $a \mathop{\smile\!\!\!\!\!\mid}_M^u b$ but $b \not\mathop{\smile\!\!\!\!\!\mid}_M^d a$.*

Proof. So assume that $p(x)$ is not co-simple over A, then there is an $L(A)$-formula $\phi(x,y)$ and $(a_\eta)_{\eta\in\omega^{<\omega}} \subseteq p(\mathbb{M})$ witnessing the tree property. Let T^{Sk} be a Skolemization of T, then of course $\phi(x,y)$ and a_η still witness the tree property. As in the proof of (5)⇒(1) in Definition 66, working in the sense of T^{Sk}, we can find an Ab-indiscernible sequence $(a_i)_{i<\omega+1}$ in $p(x)$ such that $\phi(x,a_i)$ divides over $Aa_{<i}$ and $b \models \{\phi(x,a_i)\}_{i<\omega+1}$. Let $I = (a_i)_{i<\omega}$ and $\mathrm{Sk}(AI) = M \models T$. It follows

that $a_\omega \mathop{\smile\!\!\!\!\!\!\;|}^u_M b$ (by indiscernibility) and that $b \not\mathop{\smile\!\!\!\!\!\!\;|}^d_M a_\omega$ (as $M \in \mathrm{acl}(Aa_{<\omega})$) — also the sense of T, as wanted. □

Theorem 76. *Let T be* NTP$_2$*, A an arbitrary set and assume that $p(x)$ over A is simple. Then $p(x)$ is co-simple over A.*

Proof. If $p(x)$ over A is not co-simple over A, then by Lemma 75 we find some $M \supseteq A$, $a \models p$ and b such that $a \mathop{\smile\!\!\!\!\!\!\;|}^u_M b$, but $b \not\mathop{\smile\!\!\!\!\!\!\;|}^d_M a$. As M is an extension base, it follows by Proposition 74 that $\mathrm{tp}(a/M)$ is not simple, thus $p(x)$ is not simple by Remark 61(1) — a contradiction. □

Corollary 77. *Let T be NTP$_2$ and $p(x) \in S(A)$ simple.*

(1) *If $a \models p(x)$ then $a \mathop{\smile\!\!\!\!\!\!\;|}_A b \Leftrightarrow b \mathop{\smile\!\!\!\!\!\!\;|}_A a$*
(2) *Right transitivity: If $a \models p(x)$, $B \supseteq A$, $a \mathop{\smile\!\!\!\!\!\!\;|}_A B$ and $a \mathop{\smile\!\!\!\!\!\!\;|}_B C$ then $a \mathop{\smile\!\!\!\!\!\!\;|}_A C$.*

5.3. **Independence and co-independence theorems.**

In [Kim01] Kim demonstrates that if T has TP$_1$, then the independence theorem fails for types over models, assuming the existence of a large cardinal. We give a proof of a localized and a dual versions, showing in particular that the large cardinal assumption is not needed.

Definition 78. Let $p(x)$ be (partial) type over A.

(1) We say that $p(x)$ *satisfies the independence theorem* if for any $b_1 \mathop{\smile\!\!\!\!\!\!\;|}^f_A b_2$ and $c_1 \equiv^{\mathrm{Lstp}}_A c_2 \subseteq p(\mathbb{M})$ such that $c_1 \mathop{\smile\!\!\!\!\!\!\;|}^f_A b_1$ and $c_2 \mathop{\smile\!\!\!\!\!\!\;|}^f_A b_2$, there is some $c \mathop{\smile\!\!\!\!\!\!\;|}^f_A b_1b_2$ such that $c \equiv_{b_1A} c_1$ and $c \equiv_{b_2A} c_2$.
(2) We say that $p(x)$ *satisfies the co-independence theorem* if for any $b_1 \mathop{\smile\!\!\!\!\!\!\;|}^f_A b_2$ and $c_1 \equiv^{\mathrm{Lstp}}_A c_2 \models p$ such that $b_1 \mathop{\smile\!\!\!\!\!\!\;|}^f_A c_1$ and $b_2 \mathop{\smile\!\!\!\!\!\!\;|}^f_A c_2$, there is some $c \models p$ such that $b_1b_2 \mathop{\smile\!\!\!\!\!\!\;|}^f_A c$ and $c \equiv_{Ab_1} c_1$, $c \equiv_{Ab_2} c_2$.

Of course, both the independence and the co-independence theorems hold in simple theories, but none of them characterizes simplicity.

Proposition 79. *Let T be NTP$_2$ and $p(x)$ is a partial type over A.*

(1) *If every $p'(x) \supseteq p$ with $p'(x) \in S(M)$, $M \supseteq A$ satisfies the co-independence theorem, then it is simple.*
(2) *If $p(x)$ satisfies the independence theorem, then it is co-simple.*

Proof. (1) Without loss of generality $A = \emptyset$. Assume that p is not simple, then by Fact 26 there are some formula $\phi(x, y)$, $(a_\eta)_{\eta \in \omega^{<\omega}}$ such that:

- $\left\{\phi(x, a_{\eta|i})\right\}_{i \in \omega} \cup p(x)$ is consistent for every $\eta \in \omega^\omega$.
- $\phi(x, a_\eta) \wedge \phi(x, a_{\eta'})$ is inconsistent for any incomparable $\eta, \eta' \in \omega^{<\omega}$.

By compactness we can find a similar tree of size κ large enough. Let T^{Sk} be some Skolemization of T, and we work in the sense of T^{Sk}.

Claim. There is a sequence $(c_i d_i)_{i\in\omega}$ satisfying:

(1) $\{\phi(x,c_i)\}_{i\in\omega}\cup p(x)$ is consistent.
(2) c_i, d_i start an infinite sequence indiscernible over $c_{<i}d_{<i}$.
(3) $\phi(x,d_i)\wedge\phi(x,d_j)$ is inconsistent for any $i\neq j\in\omega$.

Proof. Why? By induction we let $c_i = a_{s_1\ldots s_{i-1}s_i}$ and $d_i = a_{s_1\ldots s_{i-1}t_i}$ for some $s_i\neq t_i\in\kappa$ such that there is a $c_{<i}d_{<i}$-indiscernible sequence starting with $a_{s_1\ldots s_{i-1}s_i}, a_{s_1\ldots s_{i-1}t_i}$ (exists by Erdos-Rado as κ is large enough), so we get (2). We get (1) and (3) by the assumption on $(a_\eta)_{\eta\in\kappa^{<\kappa}}$. ☐

By compactness and Ramsey we can find a and $(c_i d_i)_{i\leq\omega+1}$ indiscernible over a, satisfying (1)–(3) and such that $a\models p(x)\cup\{\phi(x,c_i)\}$.

Let $M=\mathrm{Sk}(c_i d_i)_{i<\omega}$, a model of T^{Sk}. Then we have $c_{\omega+1}\mathop{\smile\!\!\!\!\!\mid}^u_M a$ and $d_\omega \mathop{\smile\!\!\!\!\!\mid}^u_M c_{\omega+1}$ by indiscernibility. As $c_\omega d_\omega$ start an M-indiscernible sequence, there is $\sigma\in Aut(\mathbb{M}/M)$ sending c_ω to d_ω. Let $a'=\sigma(a)$, then $a'\equiv^{\mathrm{Lstp}}_M a$, $d_\omega\mathop{\smile\!\!\!\!\!\mid}^u_M a'$ (as $c_\omega\mathop{\smile\!\!\!\!\!\mid}^u_M a$ by indiscernibility) and $\phi(a',d_\omega)$. But $\phi(x,c_{\omega+1})\wedge\phi(x,d_\omega)$ is inconsistent by (3)+(2) — so the co-independence theorem fails for $p'=\mathrm{tp}(a/M)$.

(2) Similar. ☐

Now we will show that in NTP$_2$ theories simple types satisfy the independence theorem over the extension bases. We will need the following fact from [BYC].

Fact 80. *Let T be* NTP$_2$ *and $M\models T$. Assume that $c\mathop{\smile\!\!\!\!\!\mid}_M ab$, $b\mathop{\smile\!\!\!\!\!\mid}_M a$, $b'\mathop{\smile\!\!\!\!\!\mid}_M a$, $b\equiv_M b'$. Then there exists $c'\mathop{\smile\!\!\!\!\!\mid}_M ab'$ and $c'b'\equiv_M cb$, $c'a\equiv_M ca$.*

Proposition 81. *Let T be* NTP$_2$ *and $p(x)$ a simple type over $M\models T$. Then it satisfies the independence theorem: assume that $e_1\mathop{\smile\!\!\!\!\!\mid}_M e_2$, $d_i\mathop{\smile\!\!\!\!\!\mid}_M e_i$, $d_1\equiv_M d_2\models p(x)$. Then there is $d\mathop{\smile\!\!\!\!\!\mid} e_1e_2$ with $d\equiv_{e_iA} d_i$.*

Proof. First we find some $e_1'\mathop{\smile\!\!\!\!\!\mid}_M d_2e_2$ and such that $e_1'd_2\equiv_M e_1d_1$ (Let $\sigma\in\mathrm{Aut}(\mathbb{M}/M)$ be such that $\sigma(d_1)=d_2$, then $\sigma(e_1)d_2\equiv_M e_1d_1$. As $e_1\mathop{\smile\!\!\!\!\!\mid}_M d_1$ by simplicity of $\mathrm{tp}(d_1/M)$, $\sigma(e_1)\mathop{\smile\!\!\!\!\!\mid} d_2$. Let e_1' realize a non-forking extension to d_2e_2). Then we also have $d_2\mathop{\smile\!\!\!\!\!\mid}_M e_1'e_2$ (by transitivity and symmetry using simplicity of $\mathrm{tp}(d_2/M)$).

Applying Fact 80 with $a=e_2, b=e_1', b'=e_1, c=d_2$ we find some $d\mathop{\smile\!\!\!\!\!\mid}_M e_1e_2$, $de_1\equiv_M d_2e_1'\equiv_M d_1e_1$ and $de_2\equiv d_2e_2$ — as wanted. ☐

We conclude with the main theorem of the section.

Theorem 82. *Let T be* NTP$_2$ *and $p(x)$ a partial type over A. Then the following are equivalent:*

(1) *$p(x)$ is simple (in the sense of Definition 60).*

(2) *For any $B \supseteq A$, $a \models p$ and b, $a \mathop{\smile\!\!\!\!\!\mid}^{f}_{A} b$ if and only if $b \mathop{\smile\!\!\!\!\!\mid}^{f}_{A} a$.*

(3) *Every extension $p'(x) \supseteq p(x)$ to a model $M \supseteq A$ satisfies the co-independence theorem.*

Proof. (1) is equivalent to (2) is by Definitions 60 and Corollary 77.

(1) implies (3): By Proposition 81 and Corollary 77.

(3) implies (1) is by Proposition 79. □

Problem 83. Is every co-simple type simple in an NTP_2 theory?

We point out that at least every co-simple *stably embedded* type (defined over a small set) is simple. Recall that a partial type $p(x)$ defined over A is called stably embedded if for any $\phi(\bar{x}, c)$ there is some $\psi(\bar{x}, y) \in L(A)$ and $d \in p(\mathbb{M})$ such that $p(\mathbb{M})^n \cap \phi(\bar{x}, c) = p(\mathbb{M})^n \cap \psi(\bar{x}, d)$. If $p(x)$ happens to be defined by finitely many formulas, it is easy to see by compactness that $\psi(\bar{x}, y)$ can be chosen to depend just on $\phi(\bar{x}, y)$, and not on c. But for an arbitrary type this is not true.

Proposition 84. *Let T be* NTP_2*. Let $p(x)$ be a co-simple type over A and assume that p is stably embedded. Then $p(x)$ is simple.*

Proof. Assume $p(x)$ is not simple, and let $(a_\eta)_{\eta \in \omega^{<\omega}}$, k and $\phi(x, y)$ witness this. We may assume in addition that (a_η) is an indiscernible tree over A (that is, ss-indiscernible in the terminology of [KKS12], see Definition 3.7 and the proof of Theorem 6.6 there).

By the stable embeddedness assumption, there is some $\psi(x, z) \in L(A)$ and $b \subseteq p(\mathbb{M})$ such that $\psi(x, b) \cap p(\mathbb{M}) = \phi(x, a_\emptyset) \cap p(\mathbb{M})$. It follows by the indiscernibility over A that for every $\eta \in \omega^{<\omega}$ there is $b_\eta \subseteq p(\mathbb{M})$ satisfying $\psi(x, b_\eta) \cap p(\mathbb{M}) = \phi(x, a_\eta) \cap p(\mathbb{M})$.

As $\{\phi(x, a_{\emptyset i})\}_{i<\omega}$ is k-inconsistent, it follows that $\{\psi(x, b_{\emptyset i})\}_{i<\omega} \cup p(x)$ is k-inconsistent, thus $\{\psi(x, b_{\emptyset i})\}_{i<\omega} \cup \{\chi(x)\}$ is k-inconsistent for some $\chi(x) \in p$ by compactness and indiscernibility. Again by the indiscernibility over A we have that $\{\psi(x, b_{\eta i})\}_{i<\omega} \cup \{\chi(x)\}$ is k-inconsistent for every $\eta \in \omega^{<\omega}$. It is now easy to see that $\psi'(x, z) = \psi(x, z) \wedge \chi(x)$ and $(b_\eta)_{\eta \in \omega^{<\omega}}$ witness that $p(x)$ is not co-simple over A. □

Remark 85. If $p(x)$ is actually a definable set, the argument works in an arbitrary theory since instead of extracting a sufficiently indiscernible tree (which seems to require NTP_2), we just use the uniformity of stable embeddedness given by compactness.

6. Examples

In this section we present some examples of NTP_2 theories. But first we state a general lemma which may sometimes simplify checking NTP_2 in particular examples.

Lemma 86.

(1) *If* $(\bar{a}_\alpha, \phi_{\alpha,0}(x, y_{\alpha,0}) \vee \phi_{\alpha,1}(x, y_{\alpha,1}),\ k_\alpha)_{\alpha<\kappa}$ *is an* inp*-pattern, then* $(\bar{a}_\alpha, \phi_{\alpha,f(\alpha)}(x, y_{\alpha,f(\alpha)}),$ $k_\alpha)_{\alpha<\kappa}$ *is an* inp*-pattern for some* $f : \kappa \to \{0,1\}$.

(2) *Let* $(\bar{a}_\alpha, \phi_\alpha(x, y_\alpha), k_\alpha)_{\alpha<\kappa}$ *be an* inp*-pattern and assume that* $\phi_\alpha(x, a_{\alpha 0}) \leftrightarrow \psi_\alpha(x, b_\alpha)$ *for* $\alpha < \kappa$. *Then there is an* inp*-pattern of the form* $\left(\bar{b}_\alpha, \psi_\alpha(x, z_\alpha), k_\alpha\right)_{\alpha<\kappa}$.

6.1. **Adding a generic predicate.** Let T be a first-order theory in the language L. For $S(x) \in L$ we let $L_P = L \cup \{P(x)\}$ and $T^0_{P,S} = T \cup \{\forall x\,(P(x) \to S(x))\}$.

Fact 87. [CP98] *Let T be a theory eliminating quantifiers and $\exists^\infty$. Then:*

(1) $T^0_{P,S}$ *has a model companion* $T_{P,S}$, *which is axiomatized by* T *together with*

$$\forall \bar{z}\left[\exists \bar{x}\phi(\bar{x},\bar{z}) \wedge (\bar{x} \cap \mathrm{acl}_L(\bar{z}) = \emptyset) \wedge \bigwedge\nolimits_{i<n} S(x_i) \wedge \bigwedge\nolimits_{i\neq j<n} x_i \neq x_j\right] \to$$
$$\left[\exists \bar{x}\phi(\bar{x},\bar{z}) \wedge \bigwedge\nolimits_{i\in I} P(x_i) \wedge \bigwedge\nolimits_{i\notin I} \neg P(x_i)\right]$$

for every formula $\phi(\bar{x},\bar{z}) \in L$, $\bar{x} = x_0 ... x_{n-1}$ *and every* $I \subseteq n$. *It is possible to write it in first-order due to the elimination of* $\exists^\infty$.

(2) $\mathrm{acl}_L(a) = \mathrm{acl}_{L_P}(a)$

(3) $a \equiv^{L_P} b \Leftrightarrow$ *there is an isomorphism between* L_P *structures* $f : \mathrm{acl}(a) \to \mathrm{acl}(b)$ *such that* $f(a) = b$.

(4) *Modulo* $T_{P,S}$, *every formula* $\psi(\bar{x})$ *is equivalent to a disjunction of formulas of the form* $\exists \bar{z}\phi(\bar{x},\bar{z})$ *where* $\phi(\bar{x},\bar{z})$ *is a quantifier-free* L_P *formula and for any* $\bar{a},\bar{b}$, *if* $\models \phi(\bar{a},\bar{b})$, *then* $\bar{b} \in \mathrm{acl}(\bar{a})$.

Theorem 88. *Let T be geometric (that is, the algebraic closure satisfies the exchange property, and T eliminates $\exists^\infty$) and* NTP$_2$. *Then T_P is* NTP$_2$.

Proof. Denote $a \mathop{\smile\!\!\!\!\!\mid}\nolimits^a_c b \Leftrightarrow a \notin \mathrm{acl}(bc) \setminus \mathrm{acl}(c)$. As T is geometric, $\mathop{\smile\!\!\!\!\!\mid}\nolimits^a$ is a symmetric notion of independence, which we will be using freely from now on.

Let $(\bar{a}_i, \phi(x,y), k)_{i<\omega}$ be an inp-pattern, such that $(\bar{a}_i)_{i<\omega}$ is an indiscernible sequence and $\bar{a}_i$'s are mutually indiscernible in the sense of L_P, and ϕ an L_P-formula.

Claim. For any i, $\{a_{ij}\}_{j<\omega}$ is an $\mathop{\smile\!\!\!\!\!\mid}\nolimits^a$-independent set (over $\emptyset$) and $a_{ij} \notin \mathrm{acl}(\emptyset)$.

Proof. By indiscernibility and compactness. □

Let $A = \bigcup_{i<\omega} \bar{a}_i$.

Claim. There is an infinite A-indiscernible sequence $(b_t)_{t<\omega}$ such that $b_t \models \{\phi(x, a_{i0})\}_{i<\omega}$ for all $t < \omega$.

Proof. First, there are infinitely many different b_t's realizing $\{\phi(x, a_{i0})\}_{i<\omega}$, as $\{\phi(x, a_{i0})\}_{0<i<\omega} \cup \{\phi(x, a_{0j})\}$ is consistent for any $j < \omega$ and $\{\phi(x, a_{0j})\}_{j<\omega}$ is k-inconsistent. Extract an A-indiscernible sequence from it. □

Let $p_i(x, a_{i0}) = \mathrm{tp}_L(b_0/a_{i0})$.

Claim. For some/every $i < \omega$, there is $b \models \bigcup_{j<\omega} p_i(x, a_{ij})$ such that in addition $b \notin \mathrm{acl}(A)$.

Proof. For any $N < \omega$, let

$$q_i^N(x_0...x_{N-1}, a_{i0}) = \bigcup_{n<N} p_i(x_n, a_{i0}) \cup \{x_{n_1} \neq x_{n_2}\}_{n_1 \neq n_2 < N}$$

As $b_0...b_{N-1} \models \bigcup_{i<\omega} q_i^N(x_0...x_{N-1}, a_{i0})$ and T is NTP_2, there must be some $i < \omega$ such that $\bigcup_{j<\omega} q_i^N(x_0...x_{N-1}, a_{ij})$ is consistent for arbitrary large N (and by indiscernibility this holds for every i). Then by compactness we can find $b \models \bigcup_{j<\omega} p_i(x, a_{ij})$ such that in addition $b \notin \mathrm{acl}(A)$. □

Work with this fixed i. Notice that $b_0 a_{i0} \equiv^L b a_{ij}$ for all $j \in \omega$.

Claim. The following is easy to check using that $\mathop{\smile\!\!\!\!\!\mid}^a$ satisfies exchange.

1. $\mathrm{acl}(A) \cap \mathrm{acl}(a_{ij} b) = \mathrm{acl}(a_{ij})$.
2. $\mathrm{acl}(a_{ij} b) \cap \mathrm{acl}(a_{ik} b) = \mathrm{acl}(b)$ for $j \neq k$.

Now we conclude as in the proof of [CP98, Theorem 2.7]. That is, we are given a coloring P on $\bar{a}_i$. Extend it to a P_i-coloring on $\mathrm{acl}(a_{ij} b)$ such that $a_{ij} b$ realizes $\mathrm{tp}_{L_P}(a_{i0} b_0)$, and by the claim all P_i's are consistent. Thus there is some b' such that $b_0 a_{i0} \equiv^{L_P} b' a_{ij}$ for all $j \in \omega$, in particular $b' \models \{\phi_i(x, a_{ij})\}$ — a contradiction. □

Example 89. Adding a (directed) random graph to an o-minimal theory is NTP_2.

Problem 90. Is it true without assuming exchange for the algebraic closure?

6.2. **Valued fields.** In this section we are going to prove the following theorem:

Theorem 91. *Let $\bar{K} = (K, \Gamma, k, v : K \to \Gamma, ac : K \to k)$ be a Henselian valued field of characteristic $(0,0)$ in the Denef-Pas language. Let $\kappa = \kappa^1_{\mathrm{inp}}(k) \times \kappa^1_{\mathrm{inp}}(\Gamma)$. Then $\kappa^1_{\mathrm{inp}}(K) < R(\kappa + 2, \Delta)$ for some finite set of formulas Δ (see Definition 4). In particular:*

1. *If k is NTP_2, then $\bar{K}$ is NTP_2 (as every ordered abelian group is NIP by [GS84], thus $\kappa_{\mathrm{inp}}(\Gamma) < \infty$ and NTP_2 follows by Lemma 16).*
2. *If k and Γ are strong (of finite burden), then $\bar{K}$ is strong (resp. of finite burden).*

The "in particular" part follows by 14.

Example 92. (1) Hahn series over pseudo-finite fields are NTP_2.

(2) In particular, let $K = \prod_{p \text{ prime}} \mathbb{Q}_p/\mathfrak{U}$ with $\mathfrak{U}$ a non-principal ultra-filter. Then k is pseudo-finite, so has IP by [Dur80]. And Γ has SOP of course. It is known that the valuation rings of $\mathbb{Q}_p$ are definable in the pure field language uniformly in p (see e.g. [Ax65]), thus the valuation ring is definable in K in the pure field language, so K has both IP and SOP in the pure field language. By Theorem 91 it is strong of finite burden, even in the larger Denef-Pas language. Notice, however, that the burden of K is at least 2 (witnessed by the formulas "$ac(x) = y$", "$v(x) = y$" and infinite sequences of different elements in k and Γ.

Corollary 93. [She05] *If k and Γ are strongly dependent, then K is strongly dependent.*

Proof. By Delon's theorem [Del81], if k is NIP, then K is NIP. Conclude by Theorem 91 and Fact 22. ☐

We start the proof with a couple of easy lemmas about the behavior of $v(x)$ and $ac(x)$ on indiscernible sequences which are easy to check.

Lemma 94. *Let $(c_i)_{i\in I}$ be indiscernible. Consider function $(i,j) \mapsto v(c_j - c_i)$ with $i < j$. It satisfies one of the following:*

(1) *It is strictly increasing depending only on i (so the sequence is pseudo-convergent).*
(2) *It is strictly decreasing depending only on j (so the sequence taken in the reverse direction is pseudo-convergent).*
(3) *It is constant (we'll call such a sequence "constant").*

Contrary to the usual terminology we do not exclude index sets with a maximal element.

Lemma 95. *Let $(c_i)_{i\in I}$ be an indiscernible pseudo-convergent sequence. Then for any a there is some $h \in \bar{I} \cup \{+\infty, -\infty\}$ (where $\bar{I}$ is the Dedekind closure of I) such that (taking c_∞ such that $I \frown c_\infty$ is indiscernible):*

For $i < h$: $v(c_\infty - c_i) < v(a - c_\infty)$,$v(a - c_i) = v(c_\infty - c_i)$ and $ac(a - c_i) = ac(c_\infty - c_i)$.
For $i > h$: $v(c_\infty - c_i) > v(a - c_\infty)$, $v(a - c_i) = v(a - c_\infty)$ and $ac(a - c_i) = ac(a - c_\infty)$.

Notice that in fact there is a finite set of formulas Δ such that these lemmas are true for Δ-indiscernible sequences. Fix it from now on, and let $\delta = R(\kappa + 2, \Delta)$ for $\kappa = \kappa_k \times \kappa_\Gamma$ with $\kappa_k = \kappa^1_{\text{inp}}(k)$ and $\kappa_\Gamma = \kappa^1_{\text{inp}}(\Gamma)$.

Lemma 96. *In K, there is no* inp-*pattern* $\left(\phi_\alpha(x, y_\alpha), \bar{d}_\alpha, k_\alpha\right)_{\alpha<\delta}$ *with mutually indiscernible rows such that x is a singleton and $\phi_\alpha(x, y_\alpha) = \chi_\alpha(v(x - y), y^\Gamma_\alpha) \wedge \rho_\alpha(ac(x - y), y^k_\alpha)$, where $\chi_\alpha \in L_\Gamma$ and $\rho_\alpha \in L_k$.*

Proof. Assume otherwise, and let $d_{\alpha i}=c_{\alpha i}d_{\alpha i}^{\Gamma}d_{\alpha i}^{k}$ where $c_{\alpha i}\in K$ corresponds to y, $d_{\alpha i}^{\Gamma}\in\Gamma$ corresponds to y_{α}^{Γ} and $d_{\alpha i}^{k}\in k$ corresponds to y_{α}^{k}. By the choice of δ, there is a Δ-indiscernible sub-sequence of $(c_{\alpha 0})_{\alpha<\delta}$ of length $\kappa+2$. Take a sub-array consisting of rows starting with these elements – it is still an inp-pattern of depth $\kappa+2$ – and replace our original array with it. Let $c_{-\infty}$ and c_{∞} be such that $c_{-\infty}\frown(c_{\alpha 0})_{\alpha<\kappa}\frown c_{\infty}$ is Δ-indiscernible and $\left(\bar{d}_{\alpha}\right)_{\alpha<\kappa}$ is a mutually indiscernible array over $c_{-\infty}c_{\infty}$ (so either find c_{∞} by compactness if κ is infinite, or just let it be $c_{\kappa-1,0}$ and replace our array by $\left(\bar{d}_{\alpha}\right)_{\alpha<\kappa-1}$). Let $a\models\{\phi_{\alpha}(x,d_{\alpha 0})\}_{\alpha<\kappa+1}$.

Case 1. $(c_{\alpha 0})$ is pseudo-convergent. Let $h\in\{-\infty\}\cup\kappa+1\cup\{\infty\}$ be as given by Lemma 95.

Case 1.1. Assume $0<h$. Then $v(a-c_{00})=v(c_{\infty}-c_{00})$, $ac(a-c_{00})=ac(c_{\infty}-c_{00})$. But then actually $c_{\infty}\models\phi(x,d_{00})$, and by indiscernibility of the array over c_{∞}, $c_{\infty}\models\{\phi(x,d_{0i})\}_{i<\omega}$ — a contradiction.

Case 1.2: Thus $v(a-c_{\alpha 0})=v(a-c_{\infty})$, $ac(a-c_{\alpha 0})=ac(a-c_{\infty})$ and $v(a-c_{\infty})<v(c_{\infty}-c_{\alpha 0})$ for all $0<\alpha<\kappa+1$.

Let $\chi'_{\alpha}(x',e_{\alpha i}^{\Gamma}):=\chi_{\alpha}(x',d_{\alpha i}^{\Gamma})\wedge x'<v(c_{\infty}-c_{\alpha i})$ with $e_{\alpha i}^{\Gamma}=d_{\alpha i}^{\Gamma}\cup v(c_{\infty}-c_{\alpha i})$. Finally, for $\alpha<\kappa_{\Gamma}$ let $f_{\alpha i}^{\Gamma}=\bigcup_{\beta<\kappa_k}e_{\kappa_k\times\alpha+\beta,i}$ and $p_{\alpha}(x',f_{\alpha i}^{\Gamma})=\left\{\chi'_{\beta}(x',e^{\Gamma}_{\kappa_k\times\alpha+\beta,i})\right\}_{\beta<\kappa_k}$. As $\left(f_{\alpha i}^{\Gamma}\right)$ is a mutually indiscernible array in Γ, $\left\{p_{\alpha}(x',f_{\alpha 0}^{\Gamma})\right\}_{\alpha<\kappa_{\Gamma}}$ is realized by $v(a-c_{\infty})$ and $\kappa_{\mathrm{inp}}^{1}(\Gamma)=\kappa_{\Gamma}$, there must be some $\alpha<\kappa_{\Gamma}$ and $a_{\Gamma}\in\Gamma$ such that (unwinding) $a_{\Gamma}\models\left\{\chi'_{\beta}(x',e^{\Gamma}_{\kappa_k\times\alpha+\beta,i})\right\}_{\beta<\kappa_k,i<\omega}$.

Analogously letting $\chi'_{\beta}(x',e_{\beta i}^{k}):=\rho_{\kappa_k\times\alpha+\beta}(x',d^{k}_{\kappa_k\times\alpha+\beta,i})$, noticing that $(e_{\beta i}^{k})_{\beta<\kappa_k,i<\omega}$ is an indiscernible array in k and $\kappa_k=\kappa_{\mathrm{inp}}(k)$, there must be some $a_{\rho}\in k$ and $\beta<\kappa_k$ such that $a_{\rho}\models\{\chi'_{\beta}(x',e_{\beta i}^{k})\}_{i<\omega}$.

Finally, take $a'\in K$ with $v(a'-c_{\infty})=a_{\Gamma}\wedge ac(a'-c_{\infty})=a_{\rho}$ and let $\gamma=\kappa_k\times\alpha+\beta$. As $a_{\Gamma}<v(c_{\infty}-c_{\gamma i})$ it follows that $v(a'-c_{\gamma i})=v(a'-c_{\infty})$ and $ac(a'-c_{\gamma i})=ac(a'-c_{\infty})$. But then $a'\models\{\phi_{\gamma}(x,d_{\gamma i})\}_{i<\omega}$ – a contradiction.

Case 2: (c_0^{α}) is decreasing - reduces to the first case by reversing the order of rows.

Case 3: (c_0^{α}) is constant.

If $v(a-c_{\alpha 0})<v(c_{\infty}-c_{\alpha 0})$ $(=v(c_{\beta 0}-c_{\alpha 0})$ for $\beta\neq\alpha)$ for some α, then $v(a-c_{\alpha 0})=v(a-c_{\beta 0})=v(a-c_{\infty})$ for any β, and $ac(a-c_{\alpha 0})=ac(a-c_{\infty})$ for all α's and it falls under case 1.2.

Next, there can be at most one α with $v(a-c_{\alpha 0})>v(c_{\infty}-c_{\alpha 0})$ (if also $v(a-c_{\beta 0})>v(c_{\infty}-c_{\beta 0})$ for some $\beta>\alpha$ then $v(c_{\beta 0}-c_{\alpha 0})=v(a-c_{\alpha 0})>v(c_{\infty}-c_{\alpha 0})$, a contradiction). Throw the corresponding row away and we are left with the case $v(a-c_{\alpha 0})=v(c_{\infty}-c_{\alpha 0})=v(a-c_{\infty})$ for all $\alpha<\kappa$. It follows by indiscernibility that $v(a-c_{\infty})=v(c_{\infty}-c_{\alpha i})$ for all α,i. Notice that it follows that $ac(a-c_{\alpha 0})\neq ac(c_{\infty}-c_{\alpha 0})$ and $ac(a-c_{\alpha 0})=ac(a-c_{\infty})+ac(c_{\infty}-c_{\alpha 0})$.

Let $\rho'_{\alpha}(x',e_{\alpha i}^{k}):=\rho_{\alpha}(x'-ac(c_{\infty}-c_{\alpha i}),d_{\alpha i}^{k})\wedge x'\neq ac(c_{\infty}-c_{\alpha i})$ with $e_{\alpha i}^{k}=d_{\alpha i}^{k}\cup ac(c_{\infty}-c_{\alpha i})$. Notice that $ac(a-c_{\infty})\models\left\{\rho'_{\alpha}(x',e_{\alpha 0}^{k})\right\}$ and that $\left(e_{\alpha i}^{k}\right)$ is a mutually indiscernible array in k. Thus there is some $\alpha<\kappa$ and $a_k\models\left\{\rho'_{\alpha}(x',e_{\alpha i}^{k})\right\}_{i<\omega}$.

Take $a' \in K$ such that $v(a'-c_\infty) = v(a-c_\infty) \wedge ac(a'-c_\infty) = a_k$. By the choice of a_k we have that $v(a'-c_\infty) = v(a-c_\infty) = v(c_\infty - c_{\alpha i})$ and that $ac(a'-c_\infty) \neq ac(c_\infty - c_{\alpha i})$, thus $v(a'-c_{\alpha i}) = v(a-c_\infty)$ and $ac(a'-c_{\alpha i}) = a_k + ac(c_\infty - c_{\alpha i})$. It follows that $a' \models \{\phi_\alpha(x, d_{\alpha i})\}_{i<\omega}$ — a contradiction. $\square$

Lemma 97. *In K, there is no* inp-pattern $\left(\phi_\alpha(x,y_\alpha), \bar{d}_\alpha, k_\alpha\right)_{\alpha<\delta}$ *such that x is a singleton and $\phi_\alpha(x,y_\alpha) = \chi_\alpha(v(x-y_1),...,v(x-y_n),y_\alpha^\Gamma) \wedge \rho_\alpha(ac(x-y_1),...,ac(x-y_n),y_\alpha^k)$, where $\chi_\alpha \in L_\Gamma$ and $\rho_\alpha \in L_k$.*

Proof. We prove it by induction on n. The base case is given by Lemma 96. So assume that we have proved it for $n-1$, and let $\left(\phi_\alpha(x,y_\alpha), \bar{d}_\alpha, k_\alpha\right)_{\alpha<\delta}$ be an inp-pattern with $\phi_\alpha(x,y_\alpha) = \chi_\alpha(v(x-y_1),...,v(x-y_n),y_\alpha^\Gamma) \wedge \rho_\alpha(ac(x-y_1),...,ac(x-y_n),y_\alpha^k)$ and $d_{\alpha i} = c_{\alpha i}^1 ... c_{\alpha i}^n d_{\alpha i}^\Gamma d_{\alpha i}^k$.

So let $a \models \{\phi_\alpha(x, d_{\alpha 0})\}_{\alpha<\delta}$. Fix some $\alpha < \delta$.

Case 1: $v(a - c_{\alpha 0}^1) < v(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Then $v(a-c_{\alpha 0}^1) = v(a-c_{\alpha 0}^n)$ and $ac(a-c_{\alpha 0}^1) = ac(a-c_{\alpha 0}^n)$. We take

$$\begin{aligned}\phi'_\alpha(x,d'_{\alpha i}) &= \left(\chi_\alpha(v(x-c_{\alpha i}^1),...,v(x-c_{\alpha i}^1),d_{\alpha i}^\Gamma) \wedge v(x-c_{\alpha 0}^1) < v(c_{\alpha i}^n - c_{\alpha i}^1)\right) \\ &\quad \wedge \rho_\alpha(ac(x-c_{\alpha i}^1),...,ac(x-c_{\alpha i}^1),d_{\alpha i}^\rho)\end{aligned}$$

and $d'_{\alpha i} = d_{\alpha i} \cup v(c_{\alpha i}^n - c_{\alpha i}^1)$.

Case 2: $v(a-c_{\alpha 0}^1) > v(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Then $v(a-c_{\alpha 0}^n) = v(c_{\alpha 0}^n - c_{\alpha 0}^1)$ and $ac(a-c_{\alpha 0}^n) = ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$. Take

$$\begin{aligned}\phi'_\alpha(x,d'_{\alpha i}) &= \left(\chi_\alpha(v(x-c_{\alpha i}^1),...,v(c_{\alpha 0}^n - c_{\alpha 0}^1),d_{\alpha i}^\Gamma) \wedge v(x-c_{\alpha 0}^1) > v(c_{\alpha i}^n - c_{\alpha i}^1)\right) \\ &\quad \wedge \rho_\alpha(ac(x-c_{\alpha i}^1),...,ac(c_{\alpha 0}^n - c_{\alpha 0}^1),d_{\alpha i}^\rho)\end{aligned}$$

and $d'_{\alpha i} = d_{\alpha i} \cup v(c_{\alpha i}^n - c_{\alpha i}^1) \cup ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Case 3: $v(a-c_{\alpha 0}^n) < v(c_{\alpha 0}^n - c_{\alpha 0}^1)$ and **Case 4**: $v(a-c_{\alpha 0}^n) > v(c_{\alpha 0}^n - c_{\alpha 0}^1)$ are symmetric to the cases 1 and 2, respectively.

Case 5: $v(a-c_{\alpha 0}^1) = v(a-c_{\alpha 0}^n) = v(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

Then $ac(a-c_{\alpha 0}^n) = ac(a-c_{\alpha 0}^1) - ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$. We take

$$\begin{aligned}\phi'_\alpha(x,d'_{\alpha i}) &= \left(\chi_\alpha(v(x-c_{\alpha i}^1),...,v(c_{\alpha 0}^n - c_{\alpha 0}^1),d_{\alpha i}^\Gamma) \wedge v(x-c_{\alpha 0}^1) = v(c_{\alpha i}^n - c_{\alpha i}^1)\right) \\ &\quad \wedge \left(\rho_\alpha(ac(x-c_{\alpha i}^1),...,ac(c_{\alpha 0}^n - c_{\alpha 0}^1),d_{\alpha i}^\rho) \wedge ac(x-c_{\alpha 0}^1) \neq ac(c_{\alpha i}^n - c_{\alpha i}^1)\right)\end{aligned}$$

and $d'_{\alpha i} = d_{\alpha i} \cup v(c_{\alpha i}^n - c_{\alpha i}^1) \cup ac(c_{\alpha 0}^n - c_{\alpha 0}^1)$.

In any case, we have that $\{\phi'_\alpha(x,d'_{\alpha i})\}_{i<\omega}$ is inconsistent, $\{\phi_\beta(x,d_{\beta,0})\}_{\beta<\alpha} \cup \{\phi'_\alpha(x,d'_{\alpha 0})\} \cup \{\phi_\beta(x,d_{\beta 0})\}_{\alpha<\beta<\delta}$ is consistent, and $\left(\bar{d}_\beta\right)_{\beta<\alpha} \cup \left\{\bar{d}'_\alpha\right\} \cup \left(\bar{d}_\beta\right)_{\alpha<\beta<\delta}$ is a mutually indiscernible

array. Doing this for all α by induction we get an inp-pattern of the same depth involving strictly less different $v(x-y_i)$'s – contradicting the inductive hypothesis. $\square$

Finally, we are ready to prove Theorem 91.

Proof. By the cell decomposition of Pas [Pas90], every formula $\phi(x,\bar{c})$ is equivalent to one of the form $\bigvee_{i<n}(\chi_i(x) \wedge \rho_i(x))$ where $\chi_i = \bigwedge \chi_j^i(v(x-c_j^i), \bar{d_j^i})$ with $\chi_j^i(x, \bar{d_j^i}) \in L(\Gamma)$ and $\rho_i = \bigwedge \rho_j^i(ac(x-c_j^i), \bar{e}_j^i)$ with $\rho_j^i(x, \bar{e}_j^i) \in L(k)$. By Lemma 86, if there is an inp-pattern of depth κ with x ranging over K, then there has to be an inp-pattern of depth κ and of the form as in Lemma 97, which is impossible. It is sufficient, as Γ and k are stably embedded with no new induced structure and are fully orthogonal. $\square$

Problem 98.

(1) Can the bound on $\kappa_{\text{inp}}(K)$ given in Theorem 91 be improved?

(2) Determine the burden of $K = \prod_{p \text{ prime}} \mathbb{Q}_p / \mathfrak{U}$ in the pure field language. In [DGL11] it is shown that each of $\mathbb{Q}_p$ is dp-minimal, so combined with Fact 22 it has burden 1. However K is not inp-minimal, as both v and ac are definable in the pure field language, and the residue field is infinite, so $\{v(x) = v_i\}$, $\{ac(x) = a_i\}$ shows that the burden is at least 2.

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